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Highlights in Lie Algebraic Methods

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Preface

A two-week Summer School on “Structures in Lie Theory, Crystals, Derived Functors, Harish-Chandra Modules, Invariants and Quivers” was held at Jacobs University Bremen during 9–22 August 2009 on both the geometric and algebraic aspects of Lie Theory. The participants were mainly from European countries with strong contingents from Germany, Russia, and Israel. Several high-level graduate courses were given, containing recent or original results on topics of particular current interest. The detailed notes of five of these lecture courses are reproduced in this volume. They not only provide a welcome reminder of the content of these courses to the participants, but enable all those who did not attend the meeting to profit from insights of leading specialists into the latest developments in some exciting fields of research. Even those participants who followed the courses attentively may profit from the details presented in this book.

Besides the presenters of the graduate courses, some younger researchers were invited to speak at the meeting and to submit a manuscript for publication. Thus the present volume also contains five original articles.

All the texts reproduced here were subject to a strict refereeing procedure befitting any mathematical journal.

The Courses

Spherical Varieties by Michel Brion, Grenoble

This course discusses and interrelates several classes of complex algebraic varieties equipped with an action of a connected algebraic group G. For example, the homogeneous varieties (in which G acts transitively), the spherical varieties (for which G is reductive and for which a Borel subgroup of G has an open orbit), the symmetric spaces G/G^{θ} (where again G is reductive, and θ is an involution of G), and the “wonderful varieties” in the sense of De Concini and Procesi, are examined in this course. Classification theorems are presented, especially in the case of complete varieties.

A significant aspect of this course is the systematic use of the notion of a log homogeneous variety X. This is a smooth G-variety equipped with a divisor D with normal crossings such that D is G-stable and the associated "logarithmic tangent bundle" is generated by the Lie algebra $\mathfrak{g}$ (acting on X via vector fields that preserve D). This condition implies that $X \setminus D$ is a single G orbit.

A local structure theorem is given for a complete log homogeneous variety X along a closed G-orbit, showing in particular that X is spherical if G is linear. Conversely, it is stated that a spherical homogeneous space G/H always admits a log homogeneous completion. Moreover, if H is its own normalizer in G, then the closure of the G-orbit of $\mathfrak{h}$ in the variety of Lie subalgebras of $\mathfrak{g}$ is a log homogeneous completion with a unique closed G-orbit.

An alternative construction of this "wonderful completion" is presented in the case of a semisimple group G with trivial center, viewed as a homogeneous space under $G \times G$. Let V be a simple G-module with a regular highest weight. Then $G \times G$ acts on the projectivization $\mathbb{P}(\operatorname{End} V)$, and the orbit through the class of the identity is isomorphic to G. By a result of De Concini and Procesi, the closure of that orbit is the desired wonderful completion of G and is independent of the choice of V.

The above result is generalized to an arbitrary symmetric space G/G^θ as follows. Assume that V is spherical (i.e., it admits a G^θ-invariant vector) and "regular" for that property. Then, as shown by De Concini and Procesi, the orbit closure of the corresponding point of $\mathbb{P}(V)$ is again a wonderful completion of G/G^θ, independent of V.

Finally, the notion of a wonderful variety is discussed. It is stated that there are only finitely wonderful varieties for a given semisimple group G, and there is an ongoing program to classify them.

Consequences of the Littelmann Path Model for the Structure of the Kashiwara $B(\infty)$ Crystal by Anthony Joseph, Weizmann Institute

This course starts with the observation that the $B(\infty)$ crystal of Kashiwara associated to a Kac–Moody algebra $\mathfrak{g}$ is given by a purely combinatorial construction. The properties of $B(\infty)$ were determined by taking a $q \to 0$ limit of highest weight modules over the quantized enveloping $U_q(\mathfrak{g})$.

The Littelmann path construction associates a crystal to a highest weight of $\mathfrak{g}$. Then $B(\infty)$ can be obtained by a limiting process on the highest weight. This gives a purely combinatorial way to analyze the structure of $B(\infty)$, which has the advantage that there is no need to assume that $\mathfrak{g}$ is symmetrizable. In particular the key properties that $B(\infty)$ is upper normal and canonically determined by a tensor product construction are established. Furthermore it is shown that $B(\infty)$ admits an involution which coincides with the Kashiwara involution in the symmetrizable case. Conversely, it is shown that Littelmann's crystals may be recovered from $B(\infty)$, but so far it is not known how to show that the resulting "highest weight crystals" satisfy tensor product decomposition without appealing to the Littelmann path model (or the Lusztig–Kashiwara theory of bases).

Character formulas are discussed, noting that there is not yet a purely combinatorial proof of the Weyl denominator formula if $\dim \mathfrak{g}$ is infinite.

Combinatorial Demazure flags are described. Here it is noted that the corresponding result for the tensor product of a Demazure module (global sections of sheaves on Schubert varieties) with a one-dimensional Demazure module is known only for semisimple $\mathfrak{g}$ (Mathieu) or if the root system is simply laced. The latter is notably by virtue of a positivity result in the multiplication of canonical basis elements due to Lusztig.

Under a positivity hypothesis, Nakashima and Zelevinsky showed that $B(\infty)$ admits an additive structure (which as yet has no module-theoretic interpretation). Their proof is reproduced, noting that this positivity hypothesis implies upper normality and so is liable to be rather difficult to establish.

Aside from the Littelmann construction, detailed proofs are given for most of these combinatorial results.

Structure and Representation Theory of Kac–Moody Superalgebras, by Vera Serganova, Berkeley

The essence of supersymmetry is the introduction of a sign rule in mathematical operations. Though definitions carry over to this situation in a seemingly innocent fashion, this idea leads to a new theory with many challenging open problems. From the mathematician's point of view, the "sign rule" leads to new notions such as supermanifold, Lie supergroup, and Lie superalgebra. The interrelated theories based on the notions of superanalysis, supergeometry, representation theory of Lie supergroups and Lie superalgebras started their rapid development in the works of Berezin, Bernstein, Kac, Kostant, Leites, and others.

In these lectures, Serganova concentrates on the theory of contragredient Lie superalgebras and their representations. She recalls Kac's classification of finite-dimensional simple Lie superalgebras, gives a brief introduction into Lie supergroups, and then proceeds to the definition of Kac–Moody Lie superalgebras. The existence of odd reflections ("odd" elements of the Weyl group) plays an important role when describing the structure of Kac–Moody Lie superalgebras. Serganova then presents the classification of finite-growth contragredient Lie superalgebras, due to Van de Leur in the symmetrizable case and to Hoyt and Serganova in the non-symmetrizable case.

The next topic discussed is integrable highest weight modules. In contrast with the case of contragredient Lie superalgebras, only a few finite-growth contragredient Lie superalgebras admit nontrivial highest weight modules; see Theorem 5 due to Kac and Wakimoto. Serganova proves the Weyl character formula for typical integrable highest weight modules. This formula goes back to Kac in the finite-dimensional case. (A related result, not discussed in the lectures, is Gorelik's recent proof of the "super" denominator identity–editorial note.)

In the final Sect. 4, Serganova concentrates on the case of a finite-dimensional contragredient superalgebra. Here she discusses the case of atypical finite-dimensional simple modules and, in particular, recalls the Kac–Wakimoto conjecture on the superdimension of such a module.

A point worthy of special attention is the discussion of geometric methods: the notion of an "odd associated variety" due to Duflo and Serganova, and the older Bott–Borel–Weil approach to flag supervarieties going back to Penkov.

The lectures are concluded by a list of current open problems.

Categories of Harish-Chandra Modules by Wolfgang Soergel, Freiburg

Recall that the Kazhdan–Lusztig polynomials evaluated at $q = 1$ give the Jordan–Hölder multiplicity $[M : L]$ of a simple quotient L in a Verma module M. However the polynomials themselves have two possible interpretations, either as describing the more precise data encoded as the dimensions of the extension groups $\text{Ext}^k(M, L)$ or as the Jordan–Hölder multiplicities $[M_k, L]$ with M_k the kth graded component of M. Here the resulting matrices are not the same but related to one another by matrix inversion and appropriate sign changes. (This was first suggested in O. Gabber and A. Joseph, *Ann. Sci. École Norm. Sup.* (4) 14 (1981), no. 3, 261–302, where the gradation was to be given by the Jantzen filtration. Notably the second interpretation was proven for the socle filtration in R. S. Irving, *Ann. Sci. École Norm. Sup.* (4) 21 (1988), no. 1, 47–65. It had the important consequence that one can thereby describe the Duflo involutions through the Kazhdan–Lusztig polynomials—editorial note.)

The above observation can be interpreted as an inversion formula for the Kazhdan–Lusztig matrix of polynomials. In the well known paper of A. Beilinson, V. Ginzburg, and W. Soergel, *J. AMS* 9 (1996), no. 2, pp. 473–526, this inversion formula has been given a categorical interpretation as Koszul self-duality of the category $\mathscr{O}$. In fact, Soergel sketches a proof of this result in the latter part of his lectures.

The main content of the course is a statement of a conjecture extending the above ideas to the category of Harish-Chandra modules. The starting point is the inversion formula in J. Adams, D. Barbasch, and D. A. Vogan, Progress in Mathematics, 104. Birkhaüser Boston, Inc., Boston, MA, 1992, involving two sets of matrices coming from Jordan–Hölder (JH) and intersection cohomology (IC) matrices.

Soergel's conjecture is that the category of Harish-Chandra modules is equivalent to a certain category of finite-dimensional modules determined entirely in terms of intersection cohomology. This gives the desired relationship between the JH and IC matrices and emphasizes the similarity between the category $\mathscr{O}$ and the category of

Harish-Chandra modules. It also shows how far-reaching the idea of Koszul duality is. Soergel's conjecture is proved so far for two cases, namely for $SL(2,\mathbb{R})$ and for complex groups.

Soergel discusses his conjecture in light of tilting modules. For example, in the $\mathscr{O}$ category tilting modules interpolate between projectives and simples, and can be used to establish the Bernstein–Gelfand–Gelfand reciprocity.

Generalized Harish-Chandra Modules by Gregg Zuckerman, Yale

A module M over a Lie algebra $\mathfrak{g}$ is defined to be "generalized Harish-Chandra" if there exists a Lie subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ such that M is locally finite as an $\mathfrak{l}$-module and has finite multiplicities as an $\mathfrak{l}$-module in an appropriate sense (see Definition 1.8 in the lectures). If $\mathfrak{g}$ is finite-dimensional and semisimple, $\mathfrak{l}$ is the fixed point of an involution θ of $\mathfrak{g}$, and M is finitely generated over $\mathfrak{g}$, then this reduces to the usual definition of a Harish-Chandra module.

Harish-Chandra modules first arose in the description of unitary representations of real Lie groups and have been intensively studied since the monumental work of Harish-Chandra. Notably Langlands gave a classification of simple Harish-Chandra modules. Key questions are to determine the precise multiplicities as an $\mathfrak{l}$-module and to determine when (in terms of the Langlands parameters) a simple Harish-Chandra module is unitarizable (still an open question!).

Recently there has been considerable interest in extending this well-developed theory to generalized Harish-Chandra modules, for which the course provides an introduction. A first case of (genuinely) generalized Harish-Chandra modules occurs when $\mathfrak{g}$ is semisimple and $\mathfrak{l}$ is a Cartan subalgebra. The classification of the simple modules is basically due to Fernando and Mathieu. An important fact is that the set $\mathfrak{g}[M]$ of elements of a finite-dimensional Lie algebra $\mathfrak{g}$ acting locally finitely on a $\mathfrak{g}$-module M forms a Lie subalgebra. (This assertion generally fails for infinite-dimensional Lie algebras—editorial note.)

Considerable emphasis in these lectures is put on the use of the Zuckerman functor, which roughly is the right derived functor of the functor of passing to the submodule of $\mathfrak{l}$-finite vectors. An important procedure is cohomological induction, where the Zuckerman functor is applied to a parabolically induced or produced module. Some general properties of the Zuckerman functor are described, for example, commutation with the tensor product by a finite-dimensional $\mathfrak{g}$-module and with the action of the center of $U(\mathfrak{g})$. These properties are crucial to the "translation principle." A finite multiplicity theorem is proved, which allows Zuckerman to apply his functors to the construction of generalized Harish-Chandra modules. Some information on multiplicities is provided by standard homological arguments, for example, Frobenius reciprocity and the Euler principle. In the last section Zuckerman describes his joint work with Penkov classifying simple generalized Harish-Chandra modules with "generic" minimal $\mathfrak{l}$-type when $\mathfrak{g}$ is semisimple and $\mathfrak{l}$ is reductive in $\mathfrak{g}$. (In the case of simple Harish-Chandra modules, the notion of minimal $\mathfrak{l}$-type and its role in the classification theory was extensively studied in D. A. Vogan, *Proc. Nat. Acad. Sci. USA* 74 (1977), no. 7, 2649–2650—editorial note.)

Finally, it should be noted that generalized Harish-Chandra modules have been studied also by geometric methods (D-modules). In particular, geometric methods have been helpful in the computation by Penkov, Serganova, and Zuckerman of the Lie subalgebra $\mathfrak{g}[M]$ for certain M, as well as in the recent construction of bounded multiplicity $(\mathfrak{g}, \mathfrak{l})$-modules by Penkov and Serganova.

The Papers

B-Orbits of 2-Nilpotent Matrices and Generalizations, by Magdalena Boos and Markus Reineke, Wuppertal

Let B be the standard Borel subgroup of $GL(n, \mathbb{C})$. Melnikov showed that the set of B-orbits in the set of upper triangular matrices of square zero is finite, classified them in terms of involutions in the symmetric group S_n, and showed that the inclusion relations of their closures can be defined by link patterns. The authors obtain similar results for (the larger family of) B-orbits in the set of all matrices of square zero. Their methods are quite different to those of Melnikov. Indeed, the authors use a relation of indecomposable representations of quivers to these orbits. The former are classified using the Auslander–Reiten quiver. They use results of Zwara to translate the condition of inclusion under orbit closure into module theoretic terms. This gives the minimal orbit closure inclusion via *oriented* link patterns. Finally they discuss generic B-orbits in the set of all nilpotent matrices as well as describing semi-invariants for the action. Boos and Reineke suggest that these may generate all the semi-invariants.

Weyl Denominator Identity for Finite-Dimensional Lie Superalgebras, by Maria Gorelik, Weizmann Institute

The Weyl denominator identity for the algebras in the title was formulated by Kac and Wakimoto, who also proved it in some cases. The basic idea of Gorelik's proof (which is purely combinatorial) is to show that both sides of the identity are skew invariant for the Weyl group W and that each side forms just one orbit sum. However, unlike the Lie algebra case, there are several technical difficulties even to formulate the identity.

In general, one side of the identity is just a product over the positive roots as in the Lie algebra case but with sign changes corresponding to the odd roots. The second side is given by two sets of data $W^{\#}$ and S. The former is a subgroup of the Weyl group, and the latter is a certain maximal isotropic set of simple roots.

Moreover, there can be several choices of S even up to equivalence and therefore several denominator formulas.

In an extension of the above, the Weyl denominator identity for untwisted affine Lie superalgebras with nonzero dual Coxeter number has recently been established by Gorelik (Weyl denominator identity for affine Lie superalgebras with nonzero dual Coxeter number, *J. Algebra*, to appear), and the case of zero dual Coxeter number has been established by Gorelik and Reif (Denominator identity for affine Lie superalgebras with zero dual Coxeter number, arXiv 10125879).

Hopf Algebras and Frobenius Algebras in Finite Tensor Categories by Christoph Schweigert, Hamburg, and Jurgen Fuchs, Karlstad

This contribution reviews the importance of a canonical Hopf algebra object which exists in braided tensor categories that obey certain finiteness conditions. This Hopf algebra object has various relations to topological invariants. In particular, for any object U, the morphism space $\mathrm{Hom}(U, H)$ carries a projective representation of $SL(2,\mathbb{Z})$.

Mutation Classes of 3×3 *Generalized Cartan Matrices by Ahmet Seven, Ankara*

If B is a skew-symmetrizable matrix with integer entries, its diagram $\Gamma(B)$ is defined to be the directed graph with a directed edge from i to j whenever $B_{i,j} > 0$. The author derives a criterion, in terms of the value of the Markov constant, for the diagram of 3×3 skew-symmetrizable matrices to remain acyclic under mutation in the sense of cluster algebras.

Contractions and Polynomial Lie Algebras by Benjamin Wilson, Munich

Let $A = \mathbb{C}[t]/t^{\ell}$, and let $\mathfrak{g}$ denote a Lie algebra. Their tensor product $A \otimes \mathfrak{g}$ is again a Lie algebra and, under suitable hypotheses, possesses a highest-weight theory. A reducibility criterion for its Verma modules is derived using a contraction of a direct sum of copies of $\mathfrak{g}$.

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Part I
Courses

Spherical Varieties

Michel Brion

Abstract The aim of these lecture notes is to describe algebraic varieties on which an algebraic group acts and the orbit structure is simple. We begin by presenting fundamental results for homogeneous varieties under (possibly nonlinear) algebraic groups. Then we turn to the class of log homogeneous varieties, for which the orbits are the strata defined by a divisor with normal crossings. In particular, we discuss the close relationship between log homogeneous varieties and spherical varieties, and we survey classical examples of spherical homogeneous spaces and their equivariant completions.

Keywords Spherical varieties · (Log) homogeneous varieties · Wonderful varieties

Mathematics Subject Classification (2010) 14K05 · 14L30 · 14M17 · 14M27

Notes by R. Devyatov, D. Fratila, and V. Tsanov
of a course taught by M. Brion

1 Introduction

The present constitutes the lecture notes from a mini course at the Summer School "Structures in Lie Representation Theory" from Bremen in August 2009.

The aim of these lectures is to describe algebraic varieties on which an algebraic group acts and the orbit structure is simple. The methods used come from algebraic geometry and representation theory of Lie algebras and algebraic groups.

We begin by presenting fundamental results on homogeneous varieties under (possibly nonlinear) algebraic groups. Then we turn to the class of log homogeneous varieties, recently introduced in [7] and studied further in [8]; here the orbits are the strata defined by a divisor with normal crossings. In particular, we discuss the close relationship between log homogeneous varieties and spherical varieties, and we survey classical examples of spherical homogeneous spaces and their equivariant completions.

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2 Homogeneous Spaces

Let G be a connected algebraic group over $\mathbb{C}$, and $\mathfrak{g} = T_eG$ the Lie algebra of G.

Definition A *G-variety* is an algebraic variety X with a G-action $G \times X \to X$, $(g, x) \mapsto g \cdot x$, which is a morphism of algebraic varieties.

If X is a G-variety, then the Lie algebra of G acts by vector fields on X. If X is smooth, we denote by $\mathcal{T}_X$ the tangent sheaf, and we have a homomorphism of Lie algebras

$$\mathrm{op}_X : \mathfrak{g} \longrightarrow \Gamma(X, \mathcal{T}_X),$$

and, at the level of sheaves,

$$\underline{\mathrm{op}}_X : \mathcal{O}_X \otimes \mathfrak{g} \longrightarrow \mathcal{T}_X.$$

Example

(1) *Linear algebraic groups*: $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ closed.
(2) *Abelian varieties*, that is, complete connected algebraic groups, e.g., elliptic curves. Such groups are always commutative as will be shown below.
(3) *Adjoint action*: consider the action of G on itself by conjugation. The identity $e \in G$ is a fixed point, and so G acts on $T_eG = \mathfrak{g}$. We obtain the adjoint representation $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ whose image is called the adjoint group; its kernel is the center $Z(G)$. The differential of Ad is $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ given by $\mathrm{ad}(x)(y) = [x, y]$.

Definition A G-variety X is called *homogeneous* if G acts transitively.

Let X be a homogeneous G-variety. Choose a point $x \in X$ and consider $G_x = \mathrm{Stab}_G(x)$, the stabilizer of x in G. Then G_x is a closed subgroup of G, since this is the fiber at x of the orbit map $G \to X$, $g \mapsto g \cdot x$. Moreover, this map factors through an isomorphism of G-varieties $X \cong G/G_x$. We actually have more than that, since the coset space has a distinguished point, namely eG_x. We have an isomorphism of G-varieties with a base point $(G/G_x, eG_x) \to (X, x)$. Note that every homogeneous variety X is smooth and the morphism $\underline{\mathrm{op}}_X$ is surjective.

Lemma 2.1

(i) *Let X be a G-variety, where G acts faithfully, and $x \in X$. Then G_x is linear.*
(ii) *Let $Z(G)$ denote the center of G. Then $G/Z(G)$ is linear.*
(iii) *Abelian varieties are commutative groups.*

Proof (i) Let $\mathcal{O}_{X,x}$ denote the local ring of all rational functions on X defined at x, and $\mathfrak{m}_x$ denote its maximal ideal consisting of all elements vanishing at x. We will use the following two facts from commutative algebra:

$\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is a finite-dimensional $\mathbb{C}$-vector space for all $n \geq 1$.

Krull's Intersection Theorem: $\bigcap_n \mathfrak{m}_x^n = \{0\}$.

Now, G_x acts faithfully on the local ring $\mathcal{O}_{X,x}$ and preserves $\mathfrak{m}_x$. This induces an action of G_x on each of the quotients $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$, $n \geq 1$. Denote by K_n the kernel of the morphism $G_x \to \mathrm{GL}(\mathcal{O}_{X,x}/\mathfrak{m}_x^n)$. Then $\{K_n\}_n$ is a decreasing sequence of closed subgroups of G_x, and $\bigcap_n K_n = \{e\}$ by Krull's Intersection Theorem. Since we are dealing with Noetherian spaces, the sequence K_n must stabilize, i.e., $K_n = \{e\}$, $n \gg 1$. Thus, we have obtained a faithful action of G_x on a finite-dimensional vector space, which represents G_x as a linear algebraic group.

(ii) The image of the adjoint representation is $G/Z(G)$, a closed subgroup of $\mathrm{GL}(\mathfrak{g})$. Therefore, $G/Z(G)$ is linear algebraic.

(iii) If G is an abelian variety, then $G/Z(G)$ is both complete and affine, hence a point. □

2.1 Homogeneous Bundles

Definition Let X be a G-variety, and $p : E \to X$ a vector bundle. We say that E is *G-linearized* if G acts on E, the projection p is equivariant, and G acts "linearly on fibers," i.e., if $x \in X$ and $g \in G$, then $E_x \xrightarrow{g} E_{gx}$ is linear. We will work only with vector bundles of finite rank.

If X is homogeneous, a G-linearized vector bundle E is also called *homogeneous*. If we write $(X, x) = (G/H, eH)$, then H acts linearly on the fiber E_x. In fact, there is a one-to-one correspondence:

$$\begin{pmatrix} \text{homogeneous vector} \\ \text{bundles on } G/H \end{pmatrix} \simeq \begin{pmatrix} \text{linear representations} \\ \text{of } H \end{pmatrix}.$$

More precisely, if E is a homogeneous vector bundle, then E_x is a linear representation of H. Conversely, if V is a linear representation of H, then

$$E = G \overset{H}{\times} V := \big\{(g, v) \in G \times V\big\}/(g, v) \sim \big(gh^{-1}, hv\big)$$

is a homogeneous vector bundle on X with $E_x \cong V$.

Example

(1) The tangent bundle $T_{G/H}$ corresponds to the quotient of the H-module $\mathfrak{g}$ (where H acts via the restriction of the Ad representation) by the submodule $\mathfrak{h}$.
(2) The cotangent bundle $T^*_{G/H}$, with its sheaf of differential 1-forms $\Omega^1_{G/H}$, is associated with the module $(\mathfrak{g}/\mathfrak{h})^* = \mathfrak{h}^\perp \subseteq \mathfrak{g}$.

More generally, if Y is an H-variety, then we can form, in a similar way, a bundle $X := G \overset{H}{\times} Y$ with projection $X \to G/H$, which is G-equivariant. The fiber over eH is Y. The bundle $G \overset{H}{\times} Y$ is called a *homogeneous fiber bundle*.

Remark A fiber bundle X, as above, is always a complex space, but it is not true in general that it is an algebraic variety. However, if Y is a locally closed H-stable subvariety of the projectivization $\mathbb{P}(V)$, where V is an H-module, then X is a variety (as follows from [18, Proposition 7.1]). This holds, for instance, if Y is affine; in particular, for homogeneous vector bundles, X is always a variety.

Our next aim is to classify complete homogeneous varieties. It is well known that the automorphism group $\mathrm{Aut}(X)$ of a compact complex space is a complex Lie group (see [1, Sect. 2.3]). For any topological group G, we denote by G° the connected component of the identity element. In particular, $\mathrm{Aut}^\circ(X)$ is a complex Lie group.

Theorem 2.1 (Ramanujam [21]) *If X is a complete algebraic variety, then* $\mathrm{Aut}^\circ(X)$ *is a connected algebraic group with Lie algebra $\Gamma(X, \mathcal{T}_X)$.*

Corollary 2.2 *Let X be a complete variety. Then X is homogeneous if and only if $\mathcal{T}_X$ is generated by its global sections, i.e., if and only if $\underline{\mathrm{op}}_X : \mathcal{O}_X \otimes \Gamma(X, \mathcal{T}_X) \to \mathcal{T}_X$ is surjective.*

Proof The fact that the homogeneity of X implies the surjectivity of $\underline{\mathrm{op}}_X$ was already noted above.

For the converse, denote $G = \mathrm{Aut}^\circ(X)$. We know from Ramanujam's theorem that the Lie algebra $\mathfrak{g}$ of G is identified with $\Gamma(X, \mathcal{T}_X)$. For $x \in X$, denote by $\varphi_x : G \to X$ the orbit map: $g \mapsto g \cdot x$. We observe that the surjectivity of the differential at the origin $(d\varphi_x)_e : \mathfrak{g} \to T_x X$ is equivalent to the surjectivity of the stalk map $(\underline{\mathrm{op}}_X)_x : \Gamma(X, \mathcal{T}_X) = \mathfrak{g} \to T_x X$, which is assumed to hold. Since φ_x is equivariant with respect to G, and G is homogeneous as a G-variety (considered with the left multiplication action), it follows that $d\varphi_x$ is surjective at every point. Hence, φ_x is a submersion, and therefore $\mathrm{Im}(\varphi_x) = G \cdot x$ is open in X.

We proved that for every x, the orbit $G \cdot x$ is open in X, but since X is a variety, it follows that $G \cdot x = X$, i.e., X is homogeneous. □

Corollary 2.3 *Let X be a complete variety. Then X is an abelian variety if and only if $\mathcal{T}_X$ is a trivial bundle, i.e., if and only if $\underline{\mathrm{op}}_X$ is an isomorphism.*

Proof The fact that abelian varieties have trivial tangent bundle is clear, since algebraic groups are parallelizable.

Let us show the converse implication. From Corollary 2.2 we know that X is homogeneous and hence can be written as $X = G/H$, where $G = \mathrm{Aut}^\circ(X)$, and H is the stabilizer of a given point. Now, since the tangent bundle of X is trivial, we have $\dim(X) = \dim(\Gamma(X, \mathcal{T}_X)) = \dim(\mathfrak{g}) = \dim(G)$, and hence H is finite. Therefore G is complete, i.e., an abelian variety. Now, since H fixes a point and is a normal subgroup of G, it follows (from the homogeneity) that H acts trivially on X, and hence $H = \{e\}$. □

In what follows we will make extensive use of the following theorem of Chevalley regarding the structure of algebraic groups (see [9] for a modern proof):

Theorem 2.4 *If G is a connected algebraic group, then there exists an exact sequence of algebraic groups*

$$1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \xrightarrow{p} A \longrightarrow 1,$$

where $G_{\mathrm{aff}} \trianglelefteq G$ is an affine, closed, connected, and normal subgroup, and A is an abelian variety. Moreover, G_{aff} and A are unique.

As an easy consequence, we obtain the following lemma:

Lemma 2.2 *Any connected algebraic group G can be written as $G = G_{\mathrm{aff}} Z(G)^{\circ}$.*

Proof We have

$$G/G_{\mathrm{aff}} Z(G) = \underbrace{A/p\big(Z(G)\big)}_{\text{complete}} = \frac{G/Z(G)}{G_{\mathrm{aff}} Z(G)/Z(G)},$$

and since $G/Z(G)$ is affine (see Lemma 2.1), it follows that $G/G_{\mathrm{aff}} Z(G)$ is complete and affine. Hence, $G = G_{\mathrm{aff}} Z(G) = G_{\mathrm{aff}} Z(G)^{\circ}$. □

We also need another important result (see [13] Chap. VIII):

Theorem 2.5 (Borel's fixed point theorem) *Any connected solvable linear algebraic group that acts on a complete variety has a fixed point.*

Theorem 2.6 *Let X be a complete homogeneous variety. Then $X = A \times Y$, where A is an abelian variety, and $Y = S/P$, with S semisimple and P parabolic in S.*

Proof Let $G := \mathrm{Aut}^{\circ}(X)$. Borel's theorem implies that $Z(G)^{\circ}_{\mathrm{aff}}$ acting on X has a fixed point. This group is normal in G, and since X is homogeneous, it follows that $Z(G)^{\circ}_{\mathrm{aff}}$ is trivial. Therefore, according to Chevalley's theorem, $Z(G)^{\circ} =: A$ is an abelian variety, and $G_{\mathrm{aff}} \cap A$ is finite (since it is affine and complete).

By Lemma 2.2, the map $G_{\mathrm{aff}} \times A \to G$ defined by $(g, a) \mapsto g a^{-1}$ is a surjective morphism of algebraic groups. Its kernel is isomorphic to $G_{\mathrm{aff}} \cap A$. Thus, $G \simeq (G_{\mathrm{aff}} \times A)/K$, where K is a finite central subgroup.

The radical $R(G_{\mathrm{aff}})$ has a fixed point in X by Borel's theorem. Hence, it acts trivially, and we can suppose that G_{aff} is semisimple. Similarly, we have $Z(G_{\mathrm{aff}}) = \{e\}$, i.e., G_{aff} is adjoint. Therefore, $G_{\mathrm{aff}} \cap A = \{e\}$, i.e., $K = \{e\}$. We can conclude that $G = G_{\mathrm{aff}} \times A$.

Let $x \in X$ and consider $G_x = \mathrm{Stab}_G(x)$. From Lemma 2.1 it follows that G_x is affine and therefore $G_x^{\circ} \subseteq G_{\mathrm{aff}}$. Since G/G_x is complete, G/G_x° and $G_{\mathrm{aff}}/G_x^{\circ}$ are also complete. This implies that $G_x^{\circ} =: P$ is a parabolic subgroup in G_{aff}.

Now, consider the projection $G = G_{\text{aff}} \times A \to G_{\text{aff}}$ and its restriction $p_1 : G_x \to G_{\text{aff}}$ with kernel A_x. Since $A_x = A \cap G_x$, it follows that A_x is normal in G and therefore acts trivially. Hence, $A_x = \{e\}$. Since $[p_1(G_x) : P] < \infty$ and P is parabolic, hence connected and equal to its normalizer, we find that $p_1(G_x) = P$. We have proved that $G_x = P$. Putting all together, we get $X = G_{\text{aff}}/P \times A$. □

3 Log Homogeneous Varieties

Definition Let X be a smooth variety over $\mathbb{C}$, and D an effective reduced divisor (i.e., a finite union of subvarieties of codimension 1). We say that D has *normal crossings* if for each point $x \in X$, there exist local coordinates $t_1, \ldots, t_n$ at x such that, locally, D is given by the equation $t_1 \cdots t_r = 0$ for some $r \leq n$. More specifically, the completed local ring $\widehat{\mathcal{O}_{X,x}}$ is isomorphic to the power series ring $\mathbb{C}[[t_1, \ldots, t_n]]$, and the ideal of D is generated by $t_1 \cdots t_r$.

Definition For a pair (X, D) consisting of a smooth variety and a divisor with normal crossings, we define the *sheaf of logarithmic vector fields*:

$$\mathcal{T}_X(-\log D) = \left\{ \begin{array}{l} \text{derivations of } \mathcal{O}_X \text{ which} \\ \text{preserve the ideal sheaf of} D \end{array} \right\} \subset \mathcal{T}_X.$$

Example Take $X = \mathbb{C}^n$ and $D = (t_1 \cdots t_r = 0)$, $r \leq n$, the union of some of the coordinate hyperplanes. Here $\mathcal{T}_X(-\log D)$ is generated, at $x = (0, \ldots, 0)$, by $t_1 \frac{\partial}{\partial t_1}, \ldots, t_r \frac{\partial}{\partial t_r}, \frac{\partial}{\partial t_{r+1}}, \ldots, \frac{\partial}{\partial t_n}$.

The sheaf $\mathcal{T}_X(-\log D)$ is locally free and hence corresponds to a vector bundle. However, it does not correspond to a subbundle of the tangent bundle, since the quotient has support D and hence is torsion. Observe that $\mathcal{T}_X(-\log D)$ restricted to $X \setminus D$ is nothing but $\mathcal{T}_{X \setminus D}$.

If we take the dual of the sheaf of logarithmic vector fields, we obtain the sheaf of rational differential 1-forms $\Omega^1_X(\log D)$ with poles of order at most 1 along D, called the *sheaf of differential forms with logarithmic poles*. From the previous example we see that $\Omega^1_X(\log D)$ is generated, at $x = (0, \ldots, 0)$, by $\frac{dt_1}{t_1}, \ldots, \frac{dt_r}{t_r}, dt_{r+1}, \ldots, dt_n$.

Now, suppose that a connected algebraic group G, with Lie algebra $\mathfrak{g}$, acts on X and preserves D. We get the map

$$\mathrm{op}_{X,D} : \mathfrak{g} \longrightarrow \Gamma\big(X, \mathcal{T}_X(-\log D)\big)$$

and its sheaf version

$$\underline{\mathrm{op}}_{X,D} : \mathcal{O}_X \otimes \mathfrak{g} \longrightarrow \mathcal{T}_X(-\log D).$$

Definition We call a pair (X, D) as above *log homogeneous under* G if $\underline{\mathrm{op}}_{X,D}$ is surjective. We call it *log parallelizable* if $\underline{\mathrm{op}}_{X,D}$ is an isomorphism.

Example

(1) Let $X = \mathbb{C}^n$, $D = (t_1 \cdots t_n = 0)$, and let $G = (\mathbb{C}^*)^n$ act on X by coordinate-wise multiplication. Then $\mathfrak{g} = \mathbb{C}^n$ acts via $(t_1 \frac{\partial}{\partial t_1}, \ldots, t_n \frac{\partial}{\partial t_n})$. Actually, in this case, $\underline{\mathrm{op}}_{X,D}$ is an isomorphism, so that (X, D) is log parallelizable.

(2) Let $X = \mathbb{P}^1$. Its automorphism group $G = \mathrm{PGL}(2)$ acts transitively, so X is homogeneous. Let B be the subgroup of G consisting of the images of the matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, and let $U \subset B$ consist of the images of the matrices of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$. Then B acts on X with two orbits, the fixed point ∞ and its complement. Moreover, $(\mathbb{P}^1, \infty)$ is log homogeneous for B.

On the other hand, U acts on $\mathbb{P}^1$, with the same orbits, but the action on $(\mathbb{P}^1, \infty)$ is not log homogeneous.

The 1-torus $\mathbb{C}^* = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ acts on $(\mathbb{P}^1, \{0, \infty\})$, which is log parallelizable under this action.

(3) Smooth toric varieties: let X be a smooth algebraic variety on which a torus $T = (\mathbb{C}^*)^n$ acts with a dense open orbit, and trivial stabilizer for points in that orbit. Thus, T can be identified with its open orbit in X. Put $D = X \setminus T$. It can be shown that D has normal crossings and the pair (X, D) is log parallelizable for the T-action. More precisely, a smooth and complete toric variety admits a covering by open T-stable subsets, each isomorphic to $\mathbb{C}^n$, where T acts by coordinate-wise multiplication. Noncomplete smooth toric varieties admit a smooth equivariant completion satisfying the above (for these facts, see [19, Sect. 1.4]).

Remark If (X, D) is log homogeneous under a group G, then $X_0 := X \setminus D$ consists of one G-orbit. Indeed, the map $\underline{\mathrm{op}}_{X_0} : \mathcal{O}_{X_0} \otimes \mathfrak{g} \longrightarrow \mathcal{T}_{X_0}$ is surjective, and the assertion follows by arguing as in the proof of Corollary 2.2. If (X, D) is log parallelizable, then the stabilizer of any point in $X \setminus D$ is finite.

3.1 Criteria for Log Homogeneity

Criterium 1 Let $X = G \overset{H}{\times} Y$ be a homogeneous fiber bundle. Then every G-stable divisor in X is of the form $D = G \overset{H}{\times} E$, with $E = D \cap Y$ an H-invariant divisor in Y. Moreover, (X, D) is log homogeneous (resp. log parallelizable) for G if and only if (Y, E) is log homogeneous (resp. log parallelizable) for H°. (The proof is easy, see [7, Proposition 2.2.1] for details.)

The second criterium formulated below uses a stratification of the divisor. Let X be a G-variety, where G is a connected algebraic group, and let D be an invariant divisor with normal crossings. A stratification of D is obtained as follows. Let

$$X_1 = D, \quad X_2 = \mathrm{Sing}(D), \quad \ldots, \quad X_m = \mathrm{Sing}(X_{m-1}), \quad \ldots.$$

Now, the strata are taken to be the connected components of $X_{m-1} \setminus X_m$. Each stratum is a smooth locally closed subvariety and is preserved by the G-action. Let S be a stratum, and $x \in S$ be a point. Let $t_1, \ldots, t_n$ be local coordinates for X around x such that D is defined by the equation $t_1 \cdots t_r = 0$. Then $X_{m-1} \setminus X_m$ is the set of points where precisely $m-1$ of the coordinates are 0; in particular, $S = (t_1 = \cdots = t_r = 0)$ has codimension r in X.

The normal space to S at x is defined by

$$N = N_{S/X,x} = T_x X / T_x S.$$

It contains the normal spaces to S at x of the various strata of codimension $r-1$; these spaces are precisely the lines

$$L_i = N_{S/(t_1=\cdots=\widehat{t_i}=\cdots=t_r=0),x},$$

and we have the decomposition

$$N = L_1 \oplus \cdots \oplus L_r.$$

The stabilizer G_x acts on $T_x X$, $T_x S$, and N. Since the divisor D is G-invariant, the connected component G_x° preserves each of the lines L_i, while the full stabilizer G_x is allowed, in addition, to permute them. Thus, we obtain a map

$$\rho_x : G_x^\circ \longrightarrow (\mathbb{C}^*)^r$$

with differential

$$d\rho_x : \mathfrak{g}_x \longrightarrow \mathbb{C}^r.$$

We can now formulate the following:

Criterium 2 The pair (X, D) is log homogeneous (resp. log parallelizable) for G if and only if each stratum S consists of a single G-orbit and for any $x \in S$, the map $d\rho_x$ is surjective (resp. bijective).

Furthermore, if these conditions hold, then there is an exact sequence

$$0 \longrightarrow \mathfrak{g}_{(x)} \longrightarrow \mathfrak{g} \longrightarrow \mathcal{T}_X(-\log D)_x \longrightarrow 0,$$

where $\mathfrak{g}_{(x)} = \ker(d\rho_x)$ is the stabilizer of the point x and all normal directions to S at that point.

Proof Since $\mathcal{T}_X(-\log D)$ preserves the ideal sheaf of S, we have a morphism

$$\mathcal{T}_X(-\log D)_{|S} \longrightarrow \mathcal{T}_S.$$

Hence, at the point x, there is a linear map

$$p : \mathcal{T}_X(-\log D)_x \longrightarrow T_x S.$$

In suitable local coordinates $t_1, \ldots, t_n$ for X around x, the map p is given by the projection

$$\operatorname{span}\left\{t_1 \frac{\partial}{\partial t_1}, \ldots, t_r \frac{\partial}{\partial t_r}, \frac{\partial}{\partial t_{r+1}}, \ldots, \frac{\partial}{\partial t_n}\right\} \longrightarrow \operatorname{span}\left\{\frac{\partial}{\partial t_{r+1}}, \ldots, \frac{\partial}{\partial t_n}\right\}.$$

So, we have an exact sequence

$$0 \longrightarrow \operatorname{span}\left\{t_1 \frac{\partial}{\partial t_1}, \ldots, t_r \frac{\partial}{\partial t_r}\right\} \longrightarrow \mathcal{T}_X(-\log D)_x \xrightarrow{p} T_x S \longrightarrow 0.$$

Observe that the composition $p \circ \mathrm{op}_{X,D} : \mathfrak{g} \longrightarrow T_x S$ equals op_S and hence yields an injective map $i_x : \mathfrak{g}/\mathfrak{g}_x \longrightarrow T_x S$. Thus, we have a commutative diagram, where the rows are exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{g}_x & \to & \mathfrak{g} & \to & \mathfrak{g}/\mathfrak{g}_x & \to & 0 \\ & & \downarrow d\rho_x & & \downarrow \mathrm{op}_{X,D} & & \downarrow i_x & & \\ 0 & \to & \operatorname{span}\{t_1 \frac{\partial}{\partial t_1}, \ldots, t_r \frac{\partial}{\partial t_r}\} & \to & \mathcal{T}_X(-\log D)_x & \to & T_x S & \to & 0 \end{array}$$

Since i_x is injective, the snake lemma implies that $\mathrm{op}_{X,D}$ is surjective if and only if $d\rho_x$ and i_x are surjective. Notice that i_x is onto exactly when the orbit $G \cdot x$ is open in S. This proves the first statement of the criterium.

The second statement follows: if the conditions hold, then we have an isomorphism $\ker(d\rho_x) \xrightarrow{\sim} \ker(\mathrm{op}_{X,D})$. This yields the desired exact sequence.[1] □

3.2 *The Albanese Morphism*[2]

Using the preceding criteria, we will classify all complete log parallelizable varieties. For this, we also need some results about the Albanese morphism that we now survey (see [22] for details).

Given a variety X, there exists a *universal morphism from X to an abelian variety*, i.e., a morphism

$$f : X \to A,$$

where A is an abelian variety, such that any morphism $g : X \to B$ where B is an abelian variety admits a factorization as $g = \varphi \circ f$ for a unique morphism (of varieties) $\varphi : A \to B$. Then, by rigidity of abelian varieties, φ is the composition of a

[1] This proof was not presented in the lectures and is taken from [7, Proposition 2.1.2].

[2] This material was not presented in the lectures and was added by M. Brion to prepare for the proof of Theorem 3.1.

group homomorphism and a translation of A. We say that $f =: \alpha_X$ is the *Albanese morphism* of X and that $A =: A(X)$ is the *Albanese variety*.

Next, consider a pointed variety (X, x), that is, a pair consisting of a variety X together with a base point $x \in X$. We may assume that $\alpha_X(x) = 0$ (the origin of the Albanese variety). Then

$$\alpha_X : X \to A(X), \quad x \mapsto 0$$

is universal among morphisms to abelian varieties that send x to the origin.

For the pointed variety (G, e), where G is a connected algebraic group, the Albanese morphism is nothing but the quotient homomorphism $p : G \to A$ given by Chevalley's theorem; in particular, $A(G) = A$. Indeed, given an abelian variety B, every morphism $G \to B$, $e \mapsto 0$ is a group homomorphism and sends G_{aff} to 0, in view of [17, Corollary 2.2, Corollary 3.9].

More generally, consider a homogeneous space $X = G/H$ with base point $x = eH/H$. Then the product $G_{\mathrm{aff}}H \subset G$ is a closed normal subgroup, independent of the choice of x, since the quotient $G/G_{\mathrm{aff}} = A$ is commutative, and hence we have $G_{\mathrm{aff}}gHg^{-1} = G_{\mathrm{aff}}H$ for all $g \in G$. Moreover, the quotient $G/G_{\mathrm{aff}}H = A/p(H)$ is an abelian variety. It follows easily that the quotient map $G/H \to G/G_{\mathrm{aff}}H$ is the Albanese morphism.

Suppose now that X is a smooth G-variety containing an open G-orbit $X_0 \cong G/H$. By Weil's extension theorem (see, e.g., [17, Theorem 3.1]) the morphism $\alpha_{X_0} : G/H \to G/G_{\mathrm{aff}}H$ extends to a unique morphism

$$\alpha_X : X \to G/G_{\mathrm{aff}}H,$$

the Albanese morphism of (X, x). Since α_{X_0} is G-equivariant, then so is α_X. This defines a fiber bundle

$$X = G \overset{G_{\mathrm{aff}}H}{\times} Y,$$

where the fiber $Y = \alpha_X^{-1}\alpha_X(x)$ is smooth; we say that α_X is the *Albanese fibration* of X. If G acts faithfully on X, then H is affine by Lemma 2.1, and hence $H^\circ \subset G_{\mathrm{aff}}$. In particular, $(G_{\mathrm{aff}}H)^\circ = G_{\mathrm{aff}}$.

Having these results at hand, we can now obtain the following characterization of log parallelizable varieties, due to Winkelmann (see [26]).

Theorem 3.1 *Let X be a smooth complete variety, and D be a divisor with normal crossings. Let $G = \mathrm{Aut}^\circ(X, D)$. Then (X, D) is log parallelizable for G if and only if G_{aff} is a torus and X is a fiber bundle of the form $X = G \overset{G_{\mathrm{aff}}}{\times} Y$, where Y is a smooth complete toric variety under G_{aff}. In this case, the map $X \to G/G_{\mathrm{aff}}$ is the Albanese morphism.*

In particular, if (X, D) is log parallelizable, then its connected automorphism group is an extension of an abelian variety by a torus.

Proof If we suppose that X has the described fibration properties, then log parallelizability follows directly from Criterium 1.

Conversely, suppose that (X, D) is log parallelizable. Then X contains an open G-orbit and hence is a fiber bundle $G \overset{G_{\mathrm{aff}}H}{\times} Y$ as above. By Criterium 1 again, it follows that $(Y, D \cap Y)$ is log parallelizable under G_{aff}. Since Y is complete, there exists $y \in Y$ such that the orbit $G_{\mathrm{aff}} \cdot y$ is closed in Y (y must necessarily belong to the divisor $D \cap Y$). Then the stabilizer $(G_{\mathrm{aff}})_y$ is a parabolic subgroup of G_{aff}, in particular, connected. From Criterium 2, it follows that $(G_{\mathrm{aff}})_y^{\circ}$ is a torus. But this implies that G_{aff} itself must be a torus. The variety Y is then a toric variety under G_{aff}.

Furthermore, since $G = G_{\mathrm{aff}} Z(G)^{\circ}$ (Lemma 2.2), it follows that the group G itself is commutative. Thus, we must have $H = \{e\}$, and finally, $X = G \overset{G_{\mathrm{aff}}}{\times} Y$. □

Example Let E be an elliptic curve. Let L be a line bundle on E of degree zero. Thus, L is of the form $\mathcal{O}_E(p - q)$ for some $p, q \in E$. Then $G := L \setminus$ (zero section) is a principal $\mathbb{C}^*$-bundle on E. In fact, G is a connected algebraic group, and we have an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G \longrightarrow E \longrightarrow 0$$

(as follows, e.g., from [17, Proposition 11.2]). Take $X = \mathbb{P}(L \oplus \mathcal{O}_E)$. Then the projection $X \longrightarrow E$ is a G-equivariant $\mathbb{P}^1$-bundle, that is, X can be written as $X = G \overset{\mathbb{C}^*}{\times} \mathbb{P}^1$. The divisor is $D = G \overset{\mathbb{C}^*}{\times} \{0, \infty\}$.

3.3 The Tits Morphism

Let $(X_0, x_0) = (G/H, eH)$ be a homogeneous space. For each $x \in X_0$, the isotropy Lie algebra is denoted by $\mathfrak{g}_x$. All these isotropy Lie algebras are conjugate to $\mathfrak{h}$ and in particular have the same dimension. Let

$$\mathcal{L} := \{\mathfrak{l} \subset \mathfrak{g} \text{ Lie subalgebra} \mid \dim \mathfrak{l} = \dim \mathfrak{h}\}$$

be the variety of Lie subalgebras of $\mathfrak{g}$. The group G acts on $\mathcal{L}$ via the adjoint action on $\mathfrak{g}$. We have a G-equivariant map

$$\begin{aligned} \tau : X_0 &\longrightarrow \mathcal{L} \\ x &\longmapsto \mathfrak{g}_x. \end{aligned}$$

This map is called the *Tits morphism*. The image of τ is

$$\tau(X_0) = G \cdot \mathfrak{h} = G/N_G(\mathfrak{h}) = G/N_G(H^{\circ}).$$

Thus, τ is a fibration with fiber

$$N_G(H^{\circ})/H = (N_G(H^{\circ})/H^{\circ})/(H/H^{\circ}).$$

Observe that $N_G(H^\circ)/H^\circ$ is an algebraic group and H/H° is a finite subgroup. Since $G = G_{\text{aff}} Z(G)^\circ$, and τ is clearly $Z(G)$-invariant, the image $\tau(X_0)$ is a unique orbit under G_{aff}. If the action of G on X_0 is faithful, then H is affine by Lemma 2.1. Thus, $H^\circ \subset G_{\text{aff}}$, and hence,

$$\tau(X_0) = G_{\text{aff}}/N_{G_{\text{aff}}}(H^\circ).$$

Now, let (X, D) be a log homogeneous variety for a group G, and take $X_0 = X \setminus D$. Then the Tits morphism defined on X_0 as above, extends to X by

$$\begin{aligned} \tau : X &\longrightarrow \mathcal{L} \\ x &\longmapsto \mathfrak{g}_{(x)}. \end{aligned}$$

Notice that the Tits morphism is constant if and only of (X, D) is log parallelizable for G.

Remark If X is a complete homogeneous variety, write $X = A \times Y$ according to Theorem 2.6. Then the Albanese and Tits morphisms are given by the two projections of this Cartesian product, respectively,

$$\alpha : X \longrightarrow A, \qquad \tau : X \longrightarrow Y.$$

4 Local Structure of Log Homogeneous Varieties

Let (X, D) be a complete log homogeneous variety for a connected linear algebraic group G. Then there are only finitely many orbits of G in X, and they form a stratification (Criterium 2). Let $Z = G \cdot z = G/G_z$ be a closed orbit through a given point z. The stabilizer G_z is then a parabolic subgroup of G. Let $R_u(G)$ and G_{red} be respectively the unipotent radical and a Levi subgroup (i.e., a maximal connected reductive subgroup) of G, so that we have the Levi decomposition

$$G = R_u(G) G_{\text{red}}.$$

Moreover, G_{red} is unique up to conjugation by an element in $R_u(G)$ (see [20, Chap. 6] for these results).

Arguing as in the proof of Theorem 2.6, we see that $R_u(G)$ fixes Z pointwise. Thus, G_{red} acts transitively on Z, and we have

$$Z = G_{\text{red}} \cdot z = G_{\text{red}}/(G_{\text{red}} \cap G_z)$$

with $G_{\text{red}} \cap G_z$ a parabolic in G_{red}. We are aiming to describe the local structure of X along Z.

More generally, let G be a connected reductive group acting on a normal variety X. Suppose that $Z \subset X$ is a complete orbit of this action. Fix a point $z \in Z$. The stabilizer G_z is a parabolic subgroup of G. Let P be an opposite parabolic, i.e.,

$L := P \cap G_z$ is a Levi subgroup of both G_z and P. Then $P \cdot z = R_u(P) \cdot z$ is an open cell in Z. In fact, the action of the unipotent radical on this orbit is simply transitive, so that $R_u(P) \cdot z \cong R_u(P)$. With this notation, we have the following:

Theorem 4.1 *There exists a subvariety $Y \subset X$ containing z, which is affine, L-stable, and such that the map*

$$\begin{aligned} \psi : R_u(P) \times Y &\longrightarrow X \\ (g, y) &\longmapsto g \cdot y \end{aligned}$$

is an open immersion. In particular, $Y \cap Z = \{z\}$.

Proof First notice that X can be replaced with any G-stable neighborhood of Z. A result of Sumihiro (see [24]) implies that such a neighborhood can be equivariantly embedded in a projective space $\mathbb{P}(V)$, where V is a G-module. We may even assume that X is the entire projective space $\mathbb{P}(V)$.

In this case, V contains an eigenvector v_λ for G_z with weight λ such that $z = [v_\lambda]$. There exists an eigenvector $f = f_{-\lambda} \in V^*$ for P with weight $-\lambda$ such that $f(v_\lambda) \neq 0$. Let $X_f = \mathbb{P}(V)_f \cong X \setminus (f = 0)$ be the localization of X along f. Our aim is to find an L-stable closed subvariety $Y \subset X_f$ such that $\psi : R_u(P) \times Y \longrightarrow X_f$ is an isomorphism. It is sufficient to construct a P-equivariant map

$$\varphi : X_f \longrightarrow P/L \cong R_u(P).$$

Then we may take $Y = \varphi^{-1}(eL)$.

Start with

$$\begin{aligned} \varphi : X_f &\longrightarrow \qquad \mathfrak{g}^* \\ [v] &\longmapsto \left(\xi \mapsto \frac{(\xi f)(v)}{f(v)} \right) \end{aligned}$$

Note that for $[v] \in X_f$ and $\xi \in \mathfrak{p}$, we have

$$\varphi[v](\xi) = \frac{(\xi f)(v)}{f(v)} = \frac{-\lambda(\xi) f(v)}{f(v)} = -\lambda(\xi).$$

Now, choose a G-invariant scalar product on $\mathfrak{g}$. This choice yields an identification $\mathfrak{g}^* \cong \mathfrak{g}$. The composition of this identifying map and φ is a P-equivariant map, still denoted by $\varphi : X_f \longrightarrow \mathfrak{g}$. Let $\zeta \in \mathfrak{g}$ be the element corresponding to $-\lambda \in \mathfrak{g}^*$. Let $\mathfrak{n}$ be the nil-radical of $\mathfrak{p}$. We have $\mathfrak{n} = \mathfrak{p}^\perp$, and hence $\varphi : X_f \longrightarrow \zeta + \mathfrak{n}$. The affine space $\zeta + \mathfrak{n}$ consists of a single P-orbit, and we have $P_\zeta = L$. Thus,

$$\varphi(X_f) = \zeta + \mathfrak{n} \cong P \cdot \zeta \cong P/L.$$

We have obtained the desired fiber bundle structure on X_f.[3] □

[3]This proof, due to Knop (see [14]), was not presented in the lectures and is taken from the notes of M. Brion.

Theorem 4.2 *Let (X, D) be a complete, log homogeneous variety under a connected affine algebraic group G. Let $G = R_u(G)G_{\mathrm{red}}$ be a Levi decomposition. Let $Z = G \cdot z$ be a closed orbit. Let P, L, Y be as in Theorem 4.1. Then $Y \cong \mathbb{C}^r$, where L acts via a surjective homomorphism to $(\mathbb{C}^*)^r$.*

Proof The tangent space T_zX is a G_z-module. The subspace T_zZ tangent to the orbit Z is a submodule. The normal space to Z at that point is

$$N = T_zX/T_zZ,$$

which is in turn a G_z-module. Put $r = \dim N$. From our Criterium 2 for log homogeneity (Sect. 3.1) we deduce that $G_z = G_z^\circ$ acts on N diagonally, via a surjective homomorphism $G_z \longrightarrow (\mathbb{C}^*)^r$. So the unipotent radical $R_u(G_z)$ acts trivially, and the restriction to the Levi subgroup $L \longrightarrow (\mathbb{C}^*)^r$ is surjective as well. Let $\chi_1, \ldots, \chi_r$ be the corresponding characters of L.

Theorem 4.1 implies that we can decompose T_zX into a direct sum of L-modules as

$$T_zX = T_zZ \oplus T_zY.$$

As a consequence, there is an isomorphism of L-modules

$$T_zY \cong N.$$

It follows that L acts on T_zY diagonally, with weights $\chi_1, \ldots, \chi_r$. Now, let $\mathcal{O}(Y)$ be the coordinate ring of Y, and $\mathfrak{m}$ the maximal ideal of z. Then L acts on the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ via $-\chi_1, \ldots, -\chi_r$. The action on $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is given by the characters of the form $-k_1\chi_1 - \cdots - k_r\chi_r$ with $k_i \geq 0$ and $\sum k_i = k$. Since $\mathcal{O}(Y)$ is filtered by the powers $\mathfrak{m}^k$ and is a semisimple L-module, it follows that $\mathcal{O}(Y) \cong \mathbb{C}[t_1, \ldots, t_r]$. The coordinate t_i is taken to be an L-eigenvector in $\mathfrak{m}$ mapped to the ith basis vector in $\mathfrak{m}/\mathfrak{m}^2$, an eigenvector with character $-\chi_i$. We can conclude that $Y \cong \mathbb{C}^r$ with a diagonal action of L. ☐

Corollary 4.3 *With the notation from the above theorem, let $B \subset G_{\mathrm{red}}$ be any Borel subgroup. Then B has an open orbit in X.*

Proof Since all Borel subgroups of G_{red} are conjugate, it suffices to prove the statement for a particular one. So we can assume that $B \subset P$. Then we can write $B = R_u(P)(B \cap L)$, and $B \cap L$ is a Borel subgroup of L. We have $Z(L)^\circ \subset B \cap L$. By Theorem 4.2, $Z(L)^\circ$ has an open dense orbit in Y. By Theorem 4.1, we have an open immersion $R_u(P) \times Y \longrightarrow X$. This proves the corollary. ☐

5 Spherical Varieties and Classical Homogeneous Spaces

Let G be a connected reductive group over $\mathbb{C}$, and let X be a G-variety.

Definition X is called *spherical* if it contains an open B-orbit, where B is a Borel subgroup of G.

Definition A closed subgroup $H \subset G$ is called spherical if the homogeneous variety G/H is spherical.

Exercise Show that G/H is spherical if and only if there exists a Borel subgroup B such that the set BH is open in G if and only if $\mathfrak{g} = \mathfrak{b} + \mathfrak{h}$ for some Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$.

Recall that the Tits morphism for a homogeneous space $X = G/H$ is given by

$$\begin{aligned} \tau : X &\longrightarrow \mathcal{L} \\ x &\longmapsto \mathfrak{g}_x \end{aligned}$$

where $\mathcal{L}$ is the variety of all Lie subalgebras of $\mathfrak{g}$ (one may also consider those of fixed dimension $\dim \mathfrak{h}$ as was done before). The map is G-equivariant, and its image is isomorphic to $G/N_G(\mathfrak{h})$. Thus, τ defines a homogeneous fibration $\tau : G/H \longrightarrow G/N_G(\mathfrak{h})$.

Example

(1) Every complete log homogeneous variety under a linear algebraic group G is spherical under a Levi subgroup G_{red} (see Corollary 4.3).
(2) *Flag varieties*: Every homogeneous space $X = G/P$, where P is a parabolic subgroup, is spherical. This follows from the properties of the Bruhat decomposition. Since parabolic subgroups are self-normalizing, i.e., $P = N_G(\mathfrak{p})$, the Tits morphism is an isomorphism onto its image.
(3) All *toric varieties* are spherical. Here $G = (\mathbb{C}^*)^n = B$ is its own Borel subgroup. The Tits morphism here is constant.
(4) Let $U \subset G$ be a maximal unipotent subgroup. Let $\mathfrak{n} \subset \mathfrak{g}$ be the corresponding Lie subalgebra. Then we have the decomposition $\mathfrak{g} = \mathfrak{b}^- \oplus \mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Thus the variety G/U is spherical. It can be written as a homogeneous fiber bundle of the form

$$G/U = G \overset{B}{\times} B/U.$$

The fiber B/U is isomorphic to a maximal torus T in G (we have $B = TU$). The base space is the flag variety G/B.

Now, let Y be a complete smooth toric variety under the torus T. Then G/U is embedded in $G \overset{B}{\times} Y$, which is smooth, complete, and log homogeneous (see Criterium 1, Sect. 3.1). Thus, $G \overset{B}{\times} Y$ is spherical. The Tits morphism is the projection map $G \overset{B}{\times} Y \longrightarrow G/B$ (recall that $N_G(\mathfrak{n}) = B$).

Remark There are some other equivariant completions of G/U which are not log homogeneous. For example, if

$$G = SL_2, \qquad U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

then we have the embeddings $SL_2/U \hookrightarrow \mathbb{C}^2 = (SL_2/U) \cup \{0\} \hookrightarrow \mathbb{P}^2 = (SL_2/U) \cup \{0\} \cup \mathbb{P}^1$ (the first embedding is $gU \mapsto g \cdot e_1$), and we see that $\{0\}$ is an isolated orbit of codimension 2.

(5) *Horospherical varieties*: Suppose that we have a subgroup H satisfying $U \subset H \subset G$ for a maximal unipotent subgroup U. It is an exercise to show that the normalizer $P := N_G(H)$ is parabolic, $[P, P] \subset H$, and P/H is a torus. (For example, if $H = U$, then P is a Borel subgroup of G; moreover, $[P, P] = U$, and P/H is isomorphic to a maximal torus of G). Now P/H can be equivariantly embedded in a complete, smooth toric variety Y, and then $X = G \overset{P}{\times} Y$ is an equivariant completion of G/H with Tits morphism $\tau : X \to G/P$.
(6) *Reductive groups*: Let $X = G$, where $G \times G$ acts by $(x, y) \cdot z = xzy^{-1}$. Then $(G \times G)_e = \operatorname{diag}(G)$. Note that X is spherical, since $B^- \times B$ is a Borel subgroup of $G \times G$ whenever B, B^- are opposite Borel subgroups of G, and then $(B^- \times B) \cdot e = B^- B$ is open in G (since $\mathfrak{b}^- + \mathfrak{b} = \mathfrak{g}$).

Let G be semisimple and adjoint (i.e., $Z(G) = \{e\}$). Consider the representation $G \to \mathrm{GL}(V_\lambda)$, where V_λ is a simple G-module of highest weight λ. This defines a map $G \to \mathrm{PGL}(V_\lambda)$ that is injective for regular (dominant) λ. Let $\overline{G}$ be the closure of G in $\mathbb{P}(\mathrm{End}(V_\lambda))$. Then we have the following theorem, a reformulation in the setting of log homogeneous varieties of a result due to De Concini and Procesi (see [12]).

Theorem 5.1 *$\overline{G}$ is a smooth log homogeneous $G \times G$-variety with a unique closed orbit and is independent of the choice of λ.*

Recall that $\mathrm{End}(V_\lambda) \cong V_\lambda^* \otimes V_\lambda$ as a $G \times G$-module. Let $f_{-\lambda} \otimes v_\lambda \in V_\lambda^* \otimes V_\lambda$ be an eigenvector of $B^- \times B$ formed as the tensor product of a highest weight vector $v_\lambda \in V_\lambda$ and a corresponding functional $f_{-\lambda} \in V_\lambda^*$. Then $f_{-\lambda} \otimes v_\lambda$ is an eigenvector for $B^- \times B$, and any such eigenvector is a scalar multiple of $f_{-\lambda} \otimes v_\lambda$. Therefore, the orbit

$$(G \times G) \cdot [f_{-\lambda} \otimes v_\lambda] \subset \mathbb{P}\big(\mathrm{End}(V_\lambda)\big)$$

is the unique closed orbit in $\mathbb{P}(\mathrm{End}(V_\lambda))$ and hence in $\overline{G}$.

The main step in the proof of the remaining assertions is to obtain a precise version of the local structure Theorem 4.1 for $\overline{G}$, with $z := [f_{-\lambda} \otimes v_\lambda]$, $P := B \times B^-$, and $L := T \times T$. Specifically, there exists a $T \times T$-equivariant morphism

$$\varphi : \mathbb{C}^r \longrightarrow \overline{G}, \qquad (0, \ldots, 0) \longmapsto z,$$

where $T \times T$ acts on $\mathbb{C}^r$ via

$$(t_1, t_2) \cdot (x_1, \dots, x_r) := \big(\alpha_1\big(t_1 t_2^{-1}\big)x_1, \dots, \alpha_r\big(t_1 t_2^{-1}\big)x_r\big)$$

($\alpha_1, \dots, \alpha_r$ being the simple roots), such that the morphism

$$\psi : U \times U^- \times \mathbb{C}^r \longrightarrow \overline{G}, \qquad (g, h, x) \longmapsto (g, h) \cdot \varphi(x)$$

is an open immersion; in particular, φ is an isomorphism over its image, the subvariety Y of Theorem 4.1. Since the image of ψ meets the unique closed orbit, its translates by G form an open cover of $\overline{G}$; this implies e.g., the smoothness of $\overline{G}$.

The regularity assumption for λ cannot be omitted, as shown by the following

Example Let $G = \mathrm{PGL}_n \subset \mathbb{P}(\mathrm{Mat}_n) = X$ (so that λ is the first fundamental weight). Then $D = X \setminus G = (\det = 0)$. This is an irreducible divisor, singular along matrices of rank $\leq n-2$. Thus, (X, D) is not log homogeneous for $n \geq 3$.

We now continue with our list of examples of spherical varieties:

(7) *Symmetric spaces*: Let G be a connected reductive group, and let θ be an involutive automorphism of G. Let G^θ be the subgroup of elements fixed by θ. This is a reductive subgroup, and the homogeneous space G/G^θ is affine; it is called a *symmetric space* (see [23] that we will use as a general reference for symmetric spaces).

The involution θ of G yields an involution of G/G^θ that fixes the base point; one can show that this point is isolated in the fixed locus of θ. Since G/G^θ is homogeneous, it follows that each of its points is an isolated fixed point of an involutive automorphism; this is the original definition of a symmetric space, due to E. Cartan.

A symmetric space is spherical by the *Iwasawa decomposition* that we now recall. A parabolic subgroup $P \subset G$ is called θ-*split* if P and $\theta(P)$ are opposite. Let P be a minimal θ-split parabolic subgroup. Then $L := P \cap \theta(P)$ is a θ-stable Levi subgroup of P. In fact, the derived subgroup $[L, L]$ is contained in G^θ; as a consequence, every maximal torus $T \subset L$ is θ-stable. Thus, $T = T^\theta A$, where $A := \{t \in T \,|\, \theta(t) = t^{-1}\}^\circ$, and $T^\theta \cap A$ is finite. In fact, A is a maximal θ-*split* subtorus, i.e., a θ-stable subtorus where θ acts via the inverse map.

The Iwasawa decomposition asserts that the natural map

$$R_u(P) \times A/A^\theta \longrightarrow G/G^\theta$$

is an open immersion. Since $R_u(P)A$ is contained in a Borel subgroup of G, we see that the symmetric space G/G^θ is spherical. Another consequence is the decomposition of Lie algebras

$$\mathfrak{n}(\mathfrak{p}) \oplus \mathfrak{a} \oplus \mathfrak{g}^\theta = \mathfrak{g},$$

where $\mathfrak{n}$ denotes the nilradical (see [25, Proposition 38.2.7]).

(For instance, consider the group $G \times G$ and the automorphism θ such that $\theta(x, y) = (y, x)$. Then $(G \times G)^\theta = \mathrm{diag}(G)$.)

Next, consider a G-module V_λ containing nonzero G^θ-fixed points. Let v be such a fixed point; then we have a G-equivariant map

$$G/G^\theta \longrightarrow V_\lambda, \quad gG^\theta \longmapsto g \cdot v.$$

One can show that $\dim V_\lambda^{G^\theta}$ is either 1 or 0 (see Proposition 5.1). If it is 1, the weight λ is called *spherical*. Spherical weights form a finitely generated submonoid of the monoid of dominant weights.

Theorem 5.2 *Let G be a semisimple adjoint group, θ an involution, and λ a regular spherical weight. Then the map $G/G^\theta \to \mathbb{P}(V_\lambda)$ is injective, and the closure of its image is a smooth, log homogeneous G-variety, independent of λ and containing a unique closed orbit $G \cdot [v_\lambda] \cong G/\theta(P)$.*

This generalization of Theorem 5.1 is again a reformulation in the setting of log homogeneous varieties of a result due to De Concini and Procesi; they have also shown that the Tits morphism

$$X := \overline{G \cdot [v_\lambda]} \longrightarrow \mathcal{L}$$

is an isomorphism over its image. This yields an alternative construction of X as the closure of $G \cdot \mathfrak{g}^\theta$ in the variety of Lie subalgebras.

Proposition 5.1 *Let G be a connected reductive group, and $H \subset G$ a closed subgroup. Then H is spherical if and only if for any dominant weight λ and any character $\chi \in \mathrm{Hom}(H, \mathbb{C}^*)$, we have*

$$\dim (V_\lambda)_\chi^{(H)} \le 1,$$

where $(V_\lambda)_\chi^{(H)}$ denotes the subspace of all H-eigenvectors of weight χ.

Moreover, if H is reductive and $\dim V_\lambda^H \le 1$ for any λ, then H is spherical.

Proof It is known that the $G \times G$-module $\mathbb{C}[G]$ can be decomposed as follows (see, e.g., [25, Theorem 27.3.9])

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \text{ dominant weight}} V_\lambda^* \otimes V_\lambda.$$

The embeddings of the direct summands are given by

$$f \otimes v \longmapsto a_{f,v} = \big(g \mapsto f(gv)\big).$$

Let H be spherical and consider $v_1, v_2 \in (V_\lambda)_\chi^{(H)}$. Let B be a Borel subgroup such that BH is open in G, and choose $f \in (V_\lambda^*)^{(B)}$. Then

$$\frac{a_{f,v_2}}{a_{f,v_1}} \in \mathbb{C}(G)$$

is an invariant for the right H-action. It is also an invariant for the left B-action. Thus,

$$\frac{a_{f,v_2}}{a_{f,v_1}} \in \mathbb{C}(G)^{B\times H} = \mathbb{C}^*,$$

since $B \times H$ has an open orbit in G. Hence, there exists $t \in \mathbb{C}^*$ such that $a_{f,v_2} = ta_{f,v_1}$. Now,

$$0 = f(gv_2) - tf(gv_1) = f(gv_2 - tgv_1) = f\big(g(v_2 - tv_1)\big).$$

But V_λ is irreducible and $f \neq 0$, and hence $v_2 = tv_1$. This shows the "only if" part of the first assertion.

We now show the second assertion. Let H be reductive and such that $\dim V_\lambda^H \le 1$ for all dominant λ. By a theorem of Rosenlicht, to show that G/H contains an open B-orbit, it suffices to show that every rational B-invariant function on G/H is constant, i.e., $\mathbb{C}(G/H)^B = \mathbb{C}^*$. Since G/H is affine, $\mathbb{C}(G/H)$ is the fraction field of $\mathbb{C}[G/H]$. Let $f \in \mathbb{C}(G/H)^B$. Then the set of all "denominators" $D \in \mathbb{C}[G/H]$ such that $fD \in \mathbb{C}[G/H]$ is a nonzero B-stable subspace of $\mathbb{C}[G/H]$. Hence, this subspace contains an eigenvector of B, i.e., we may write $f = f_1/f_2$, where $f_1, f_2 \in \mathbb{C}[G/H]_\mu^{(B)} = \mathbb{C}[G]_\mu^{(B)\times H}$. Using the above decomposition of the $G \times G$-module $\mathbb{C}[G]$, it follows that

$$f_i = a_{\phi,v_i} \quad (i = 1, 2),$$

where $\phi \in (V_\lambda^*)^{(B)}$, $v_1, v_2 \in V_\lambda^H$, and $V_\lambda = V_\mu^*$. Thus, $v_2 = tv_1$, and $f = t$.

The proof in the nonreductive case relies on the same ideas; the details will not be given here. □

Proposition 5.2 *Let $H \subset G$ be a spherical subgroup, and $N_G(H)$ its normalizer. Then $N_G(H)/H$ is diagonalizable (i.e., it is isomorphic to a subgroup of some $(\mathbb{C}^*)^N$). Moreover, $N_G(H) = N_G(\mathfrak{h}) = N_G(H^\circ)$.*

Proof For any homogeneous space G/H, the quotient $N_G(H)/H$ acts on G/H on the right as follows:

$$\gamma \cdot gH = g\gamma^{-1}H = gH\gamma^{-1}.$$

This yields the isomorphism

$$N_G(H)/H = \mathrm{Aut}^G(G/H).$$

Also, note that $N_G(H) \subset N_G(H^\circ) = N_G(\mathfrak{h})$.

We now prove the first assertion in the case that H is reductive. Then the natural action of $N_G(H)/H$ on $\mathbb{C}[G/H]$ is faithful, since $\mathbb{C}(G/H)$ is the fraction field of $\mathbb{C}[G/H]$. But we have a decomposition

$$\mathbb{C}[G/H] \cong \bigoplus_{\lambda} V_\lambda^* \otimes V_\lambda^H$$

as $G \times N_G(H)/H$-modules, in view of the decomposition of $\mathbb{C}[G]$ as $G \times G$-module. Moreover, each nonzero V_λ^H is a line by Proposition 5.1. Thus, $N_G(H)/H$ acts on V_λ^H via a character χ_λ, and this yields the desired embedding $N_G(H)/H \hookrightarrow (\mathbb{C}^*)^N$.

The argument in the case of a nonreductive subgroup H follows similar lines, by replacing invariants of H with eigenvectors.

It remains to show that $N_G(H) \supset N_G(H^\circ)$. For this, observe that H° is spherical. Hence, the group $N_G(H^\circ)/H^\circ$ is diagonalizable and, in particular, commutative. So $N_G(H^\circ)/H^\circ$ normalizes H/H°, i.e., $N_G(H^\circ)$ normalizes H. □

We state without proof the following important result, with contributions by several mathematicians (among which Demazure, De Concini, Procesi, Knop, Luna) and the final step by Losev (see [16]).

Theorem 5.3 *Let G/H be a spherical homogeneous space. Then*

(1) *G/H admits a log homogeneous equivariant completion.*
(2) *If $H = N_G(H)$, then $\overline{G \cdot \mathfrak{h}} \subset \mathcal{L}$ is a log homogeneous equivariant completion with a unique closed orbit.*

Definition A *wonderful variety* is a complete log homogeneous G-variety X with a unique closed orbit.

The G-orbit structure of wonderful varieties is especially simple: the boundary divisor has the form $D = D_1 \cup \cdots \cup D_r$, with D_i irreducible and smooth. The closed orbit is $D_1 \cap \cdots \cap D_r$, and the orbit closures are precisely the partial intersections $D_{i_1} \cap \cdots \cap D_{i_s}$, where $1 \le i_1 < \cdots < i_s \le r$. In particular, r is the codimension of the closed orbit, also known as the *rank* of X.

For a wonderful variety X, the Tits morphism $\tau : X \to \mathcal{L}$ is finite. In particular, the identity component of the center of G acts trivially on X, and hence we may assume that G is semisimple.

Let us discuss some recent results and work in progress on the classification of wonderful varieties.

Theorem 5.4 *There exist only finitely many wonderful G-varieties for a given semisimple group G.*

This finiteness result, a consequence of [2, Corollary 3.2], is obtained via algebro-geometric methods (invariant Hilbert schemes) which are noneffective in nature. On the other hand, a classification program developed by Luna has been completed for many types of semisimple groups: in type A by Luna himself (see [15]), D by Bravi and Pezzini (see [3]), E by Bravi (see [4]), and F by Bravi and Luna (see [6]).

There is a geometric approach to Luna's program, initiated by Bravi and Cupit-Foutou (see [5]) via invariant Hilbert schemes, and currently developed by Cupit-Foutou (see [10, 11]). The starting point is the following geometric realization of wonderful varieties: let X be such a variety, with open orbit G/H, and let $v \in (V_\lambda)^{(H)}_\chi$, where λ and χ are regular. Then X is the normalization of the orbit closure $\overline{G \cdot [v]} \subset \mathbb{P}(V_\lambda)$.

This orbit closure may be nonnormal, as shown by the example of $\mathbb{P}^1 \times \mathbb{P}^1$ viewed as the wonderful completion of SL_2/T. If $V = V_n = \mathbb{C}[x, y]_n$ and $v = x^p y^q$, $p \neq q$, then $\overline{SL_2 \cdot [v]} \subset \mathbb{P}(V)$ is singular, but its normalization is $\mathbb{P}^1 \times \mathbb{P}^1$. (Here SL_2 acts on $\mathbb{C}[x, y]_n$ in the usual way.)

Finally, the structure of general complete log homogeneous varieties reduces to those of wonderful and of toric varieties, in the following sense. Let X be a log homogeneous equivariant completion of a spherical homogeneous space G/H, and let $\underline{X}$ be the wonderful completion of $G/N_G(H)$. Then the natural map $G/H \to G/N_G(H)$ extends (uniquely) to an equivariant surjective map $\tau : X \to \underline{X}$. Moreover, the general fibers of τ are finite disjoint unions of complete smooth toric varieties. (Indeed, τ is just the Tits morphism, and its general fibers are closures of $N_G(H)/H$, a finite disjoint union of tori). We refer to [7, Sect. 3.3] for further results on that reduction.

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Consequences of the Littelmann Path Theory for the Structure of the Kashiwara $B(\infty)$ Crystal

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Abstract These lectures give a review of results on the Kashiwara $B(\infty)$ crystal defined for a Kac–Moody Lie algebra, which can be obtained using the Littelmann path model. In this we do not need to assume, as does Kashiwara, that the Cartan matrix is symmetrizable.

First of all $B(\infty)$ is defined and shown to be upper normal. The latter is a deep result with no known easy proof. Again with respect to each simple root index i, it is shown that $B(\infty)$ has a canonical decomposition in the form $B^i \otimes B_i$, where B_i is the i^{th} elementary crystal. This allows one to establish in this greater generality the existence of an involution on $B(\infty)$, which extends the Kashiwara involution from the symmetrizable case. It is a new result.

Following earlier works of the author, the meaning of the Kashiwara involution to tensor product decomposition is described and the existence of combinatorial Demazure flags for certain tensor products, exhibited.

A paper of Nakashima and Zelevinski is reviewed. This proves an additivity property of $B(\infty)$ under a positivity hypothesis which they established in a few cases. It is noted that this positivity hypothesis immediately implies the upper normality of $B(\infty)$ and so is liable to be very difficult to establish in all generality.

Keywords Crystals · Tensor product · Demazure formula

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Preamble Crystals arose from viewing the quantum parameter q in quantized enveloping algebras as the temperature in the belief that the $q \to 0$ limit used should result in a significant simplification.

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To define a $q \to 0$ limit mathematically one needs a lattice. This can be provided by a basis which in the present instance could be Lusztig's canonical basis. Here Kashiwara found a path to the resulting lattice without needing an explicit basis.

In some sense the $q \to 0$ limit strips a module of its linear structure, so we are reduced to combinatorics. Conversely linear structure is the "Pons Asinorum" of combinatorics—a bridge over which the donkey cannot pass, or as more poetically expressed by Robbie Burns—"a running stream they dare na cross". Actually the poet was referring to wizards and witches, his comment being a tip to anyone hotly pursued by a scantily clad young female in the depths of the night [3].

Now although wizards dare not cross the bridge to linear algebra, they can look across the running stream and try to imitate what is happening on the other side. This was the role of Littelmann [28–30], in constructing a purely combinatorial theory of crystals recovering many results previously obtained using the linear structure of representation theory.

The main theme of these lectures is a purely combinatorial analysis of Kashiwara's $B(\infty)$ crystal. The latter is supposed to represent the algebra of functions on the open Bruhat cell, thus one may already anticipate that it will admit several realizations corresponding to different Bott-Samelson desingularizations which must be shown to be equivalent. Again from the standpoint of global sections on invertible sheaves, $B(\infty)$ should provide realizations of crystals corresponding to integrable highest weight modules.

A fundamental fact is that $B(\infty)$ is a "highest weight" crystal. This was first proved by Kashiwara; but his argument was neither simple nor purely combinatorial. It used the quantized enveloping algebra and so required the Cartan matrix to be symmetrizable. The Littelmann path model does not need this condition. We already gave an exposition of this work [16], so will not repeat the details here, only draw the consequences.

The first consequence is the independence of $B(\infty)$ on presentation. A second is the recovery of the Littelmann closed family of normal highest weight crystals whereby such a family is shown to be unique. A third new consequence is a purely combinatorial construction of the Kashiwara involution which is also valid in the not necessarily symmetrizable case.

$B(\infty)$ has a particularly simple character suggesting it to possess the structure of an additive semigroup. We describe a result of Nakashima and Zelevinsky [33] which attempts to give this property and notably that $B(\infty)$ is highest weight. Unfortunately they need to impose a positivity hypothesis whose range of validity is unclear. Again this additive structure is hardly ever free. It would be interesting to have an algorithm to compute generators.

A further significant fact is that the crystal operators satisfy the Coxeter relations when restricted to $B(\infty)$. This leads naturally to "Demazure crystals". They are shown to satisfy a "string property" from which their characters can be computed via the Demazure algorithm. Certain of their tensor products admit a combinatorial version of a Demazure flag. The corresponding module theoretic fact has been established for semisimple Lie algebras by Mathieu [21] and in the simply-laced case [15]. A complete proof would require a better understanding of the possible matrix elements for simple root vectors acting on the canonical/global basis and this in turn requires a better understanding of $B(\infty)$, which parameterizes the latter.

1 Introductory Survey

1.1 A crystal is a very special graph built from a Cartan matrix A of a Kac–Moody Lie algebra $\mathfrak{g}_A$. Here one fixes a countable set I and takes A to have integer entries $a_{i,j} : i, j \in I$. For A to have reasonable properties one imposes that $I = I_{\text{re}} \sqcup I_{\text{im}}$ with the conditions

(1) $a_{i,i} = 2 : i \in I_{\text{re}}, a_{i,i} \in -\mathbb{N} : i \in I_{\text{im}}$,
(2) $a_{i,j} \in -\mathbb{N}$, if $i \neq j$,
(3) $a_{i,j} \neq 0 \Leftrightarrow a_{j,i} \neq 0$.

When $I_{\text{im}} \neq \emptyset$, we call this the Borcherds case.

The Cartan matrix A is said to be symmetrizable if there exist positive integers $d_i : i \in I$ such that $\{d_i a_{i,j}\}$ is a symmetric matrix. In this case the relations in $\mathfrak{g}_A$ are known. They are customarily referred to as the Serre relations. Also in this case one may find a quantization $U_q(\mathfrak{g}_A)$ of the enveloping algebra $U(\mathfrak{g}_A)$ of $\mathfrak{g}_A$. Below we shall often omit the A subscript.

1.2 Crystal theory arose partly from Lusztig's theory of canonical bases [31]. The latter have some striking properties. For example let $\delta M(0)$ denote the $\mathscr{O}$ dual of the Verma module of highest weight 0. (If $\dim \mathfrak{g}_A < \infty$, that is if $\mathfrak{g}_A$ is semisimple, one may identify $\delta M(0)$ with the algebra of regular functions on the open Bruhat cell.) Now fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and let P^+ denote the set of dominant weights. Let $V(\lambda)$ denote the simple $U(\mathfrak{g})$ module of highest weight λ and $\mathbb{C}_{-\lambda}$ the one dimensional $\mathfrak{b}$ module of weight $-\lambda$.

A relatively easy fact [7, 2.2] is that there exists a unique $U(\mathfrak{b})$ module embedding $\varphi_\lambda : V(\lambda)|_{U(\mathfrak{b})} \otimes \mathbb{C}_{-\lambda} \hookrightarrow \delta M(0)|_{U(\mathfrak{b})}$. A much deeper fact is that $\delta M(0)$ admits a basis (the dual canonical basis) such that $\operatorname{Im} \varphi_\lambda$ is spanned by a subbasis. Therefore in particular, the $\operatorname{Im} \varphi_\lambda : \lambda \in P^+$ form a distributive lattice of subspaces [11, 6.2.20]. From this one may prove both the Kostant and Richardson separation theorems and extend both to the quantum case [2], [11, 7.3.8].

A further important fact is that a Demazure module [23], which is the space of global functions on an ample invertible sheaf over a Schubert variety, admits a basis formed from a subset of the canonical basis. Moreover this subset is described by a "Demazure crystal" which has some interesting properties (3.4.9, 3.5.2).

1.3 Crystals may be regarded as a certain $q \to 0$ limit of simple highest weight modules in which the structure of latter "crystallize" to a simpler form as the "temperature" q goes to zero. In this the Kashiwara theory [22] is quite elementary though extremely complicated. Moreover by a process reversing the $q \to 0$ limit used, called globalization, Kashiwara constructed a global basis [22] for a simple highest weight module, which Grojnowski and Lusztig [4] showed coincided with the canonical basis of Lusztig.

1.4 The Kashiwara construction was further extended to the Borcherds case by Kashiwara in collaboration with Jeong and Kang [19]. This requires A to be symmetrizable (so that $U_q(\mathfrak{g}_A)$ can be constructed).

1.5 From the above Kashiwara formulated an abstract notion of a crystal [18, 23] which may be viewed as the "skeleton" of a simple highest weight module, where notably linear structure is eliminated (so allowing many other possible crystals). In addition, Kashiwara gave a tensor structure on the set of crystals. It is associative, but not commutative. The rules used to describe tensor structure appear rather ad hoc, though they do come from the tensor structure on direct sums of simple highest weight modules through the above $q \to 0$ limit.

1.6 Littelmann [28] discovered a purely abstract construction of certain crystals in which each element is a piecewise linear path. The Kashiwara operators resulting from the simple root vectors which are generators of $U_q(\mathfrak{g}_A)$, become operations on paths. Notably concatenation of paths gives tensor product structure and remarkably the Kashiwara rules are obtained, moreover in a natural fashion.

1.7 The Littelmann theory does not require A to be symmetrizable. A key point is how to choose families of paths giving crystals corresponding to the simple integrable highest weight $U(\mathfrak{g}_A)$ modules $V(\lambda) : \lambda \in P^+$. Here Littelmann was motivated by Lakshmibai-Seshadri theory of standard monomial bases for such modules which these authors had constructed in type A and several other cases. Littelmann calls the resulting paths *LS* paths. A major success of the Littelmann path theory [29] was to obtain via Lusztig's quantum Frobenius map (which requires A to be symmetrizable) a construction of standard monomial bases (which are not unique) for simple highest weight module for $\mathfrak{g}_A$ given A symmetrizable (and not of Borcherds type).

1.8 Recently Lamprou and myself [17] extended the Littelmann path model to the Borcherds case (again without the assumption that A is symmetrizable). That this works is rather surprising because in this extension there is now less connection with the ideas which led to *LS* paths.

1.9 One now has two parallel theories of crystals modeled on the family $V(\lambda) : \lambda \in P^+$, namely that of Kashiwara and that of Littelmann. It turns out that there are two ways of proving [14, Chaps. 8, 10] an isomorphism between these two sets of crystals; but only one works in the Borcherds case. This proof involves a family $\mathbb{F}_A = \{B(\lambda) : \lambda \in P^+\}$ of so-called normal highest weight crystals closed under tensor product. Once diagonal entries of A have been fixed, the existence of such a family requires A to satisfy (2) and (3) of 1.1.

In Sect. 2.5 we prove uniqueness of the family $\mathbb{F}_A$. This involves an "abstract" version of the Kashiwara $B(\infty)$ crystal defined without the condition that A be symmetrizable.

1.10 At present there are two methods to construct $\mathbb{F}_A$. The first (valid only for A symmetrizable) is due to Kashiwara taking a $q \to 0$ limit. The second due to Littelmann is via his path model. Both are elementary but rather complicated. Here we point out (2.5.9) that to construct $\mathbb{F}_A$ we only need to know that the abstract $B(\infty)$ crystal is upper normal. However we cannot say just from this that the family is closed. (No doubt this will be possible through a little extra work. For the moment it only follows by proving that it coincides with Littelmann's family and then applying the analysis of Littelmann, [28], [11, Sect. 6.4].)

Analyzing work of Nakashima and Zelevinsky [33], we found (Remark 3 of 3.6.4) that this upper normality results in a much easier fashion from their results. Unfortunately at present they must impose a positivity hypothesis which has not been verified in general. Assuming positivity, their work also shows that $B(\infty)$ admits (several) additive semigroup structures and it would be interesting to give a meaning to these structures, as well as to give an algorithm for finding generators (unfortunately free generators hardly ever exist).

1.11 We give (2.5.13–2.5.25) a purely combinatorial construction of the Kashiwara involution on $B(\infty)$. This extends this involution to the not necessarily symmetrizable case.

1.12 Following Littelmann we show that the crystal operators acting on $\mathbb{F}_A$ or on $B(\infty)$ give rise to a singular Hecke algebra. (We suggest that this also holds for the Nakashima–Zelevinsky operators—3.6.6). This Hecke algebra leads naturally to "Demazure crystals" which model the Demazure modules discussed in 1.2. Demazure had introduced what we call a string property and he had mistakenly suggested that this holds for Demazure modules [5, 6]. From this he found a family of operators on the group algebra of the weight lattice which also generate a singular Hecke algebra. Here following Kashiwara we show (Theorem 3.4.6) that the "Demazure crystals" admit the string property and that thereby their characters are given through the Demazure operators. For the Demazure modules themselves, the Demazure character formula was first proved for most characteristics and for $\mathfrak{g}_A$ semisimple by Andersen [1] using the Steinberg module. It was later proved by Kumar [24] and Mathieu [32] in the arbitrary Kac–Moody setting but in characteristic zero. Using globalization, Kashiwara obtained the Demazure character formula in all characteristics [23]; but assuming A symmetrizable. However, the problem of determining the more precise information encoded in the characters of their $\mathfrak{n}$ homology is still open [10, 25].

1.13 Certain tensor products of Demazure modules admit a filtration whose quotients are again Demazure modules (at least for $\mathfrak{g}_A$ semisimple—Mathieu–Polo [21], or if A is simply-laced [15]). Here we note (3.5) a combinatorial version of Demazure flags, though we do not give the details of the main theorem (3.5.3) as this has been amply explained in other notes of ours [16] available on the web.

1.14 We give an interpretation of the Kashiwara involution in terms of highest weight elements in the tensor product of crystals from $\mathbb{F}_A$, reviewing 3.2.1–3.2.4 earlier work of ours [13, Sect. 4]. At first this seemed a pure crystal phenomenon. However we note in 3.2.5 that this has a direct module theoretic analogue. In principle this should relate the left and right Brylinski–Kostant filtrations [8]; but this is highly optimistic and we do not pursue the matter.

1.15 Last but not least we describe a theorem of Nakashima and Zelevinsky [33] showing that $B(\infty)$ is defined by linear inequalities and hence that it has an additive semigroup structure. At present this is not completely general and requires a positivity hypothesis. Moreover it seems entirely ad hoc having no module theoretic interpretation. Yet as noted in Remark 3 of 3.6.4 positivity implies that $B(\infty)$ is upper normal. Thus when it holds we obtain by 2.5.9 a further way to construct the family $\mathbb{F}_A$ of normal highest weight crystals, though one cannot expect that this will be enough to establish their closure property.

1.16 Further Reading Several texts on crystals have appeared. We shall mainly refer to our own notes [14, 15]. However one may also consult the book of Hong and Kang [8] devoted almost entirely to crystal theory.

2 Basic Definitions, Tensor Structure and the Uniqueness Theorem

2.1 The Weyl Group

2.1.1 Recall 1.1. Let A be a Cartan matrix with countable index set I. Unless specifically noted we assume that all the diagonal entries $a_{i,i} : i \in I$ of A are equal to 2, that is $I = I_{\text{re}}$.

2.1.2 Although not at first sight obvious, the Weyl group W plays a fundamental role in the theory of crystals, particularly in the construction of $\mathbb{F}_A$.

To define W it is useful though not essential to realize the entries $a_{i,j} : i, j \in I$ as follows. Let $\mathfrak{h}$ be a $\mathbb{Q}$ vector space admitting linearly independent elements $\alpha_i^\vee$: $i \in I$, called simple coroots, and define $\alpha_j \in \mathfrak{h}^* : j \in I$, called simple roots, to satisfy $\alpha_i^\vee(\alpha_j) = a_{i,j}, \forall i, j \in I$. Augment $\mathfrak{h}$ if necessary to ensure that the $\alpha_j : j \in I$ are also linearly independent. (If $n := |I| < \infty$, it is sufficient that $\dim \mathfrak{h} \geq 2n - \text{rk}\, A$.)

2.1.3 For all $i \in I$, define $s_i \in \text{Aut}(\mathfrak{h}^*)$ by the formula

$$s_i(\lambda) = \lambda - \alpha_i^\vee(\lambda)\alpha_i,$$

and let $W = \langle s_i : i \in I \rangle$ be the subgroup of $\text{Aut}(\mathfrak{h}^*)$ they generate. It is known that W is a Coxeter group with $S = \{s_i\}_{i \in I}$ as its set of generators. The relations in W are of two types.

First the Coxeter relations defined for each pair (i, j) of distinct elements of I, as follows. Set $m_{i,j} = a_{i,j}a_{j,i}$. The Coxeter relations take the form

$$s_i s_j = s_j s_i, \quad \text{if } m_{i,j} = 0; \qquad s_i s_j s_i = s_j s_i s_j, \quad \text{if } m_{i,j} = 1,$$
$$(s_i s_j)^{m_{i,j}} = (s_j s_i)^{m_{i,j}}, \quad \text{if } m_{i,j} = 2, 3,$$

with no relation if $m_{i,j} \geq 4$.

Second the relations $s_i^2 = 1 : i \in I$.

Given $w \in W$ we can write $w = s_{i_1} s_{i_2} \cdots s_{i_l} : i_j \in I$. It is called a reduced expression if l takes its smallest possible value, called the reduced length $l(w)$ of w.

2.1.4 There are a number of combinatorial results concerning Weyl groups which we shall review briefly. For proofs one may consult [11, Sect. A.1].

Set $\pi^\vee := \{\alpha_i^\vee\}_{i\in I}$, $\pi := \{\alpha_i\}_{i\in I}$, $\Delta_{\text{re}} := W\pi \subset \mathfrak{h}^*$, $\Delta_{\text{re}}^\pm := \Delta_{\text{re}} \cap \pm\mathbb{N}\pi$. A remarkable fact [20, Lemma 3.7] is that $\Delta_{\text{re}} = \Delta_{\text{re}}^+ \cup \Delta_{\text{re}}^-$, that is every element of Δ_{re} written as a sum of simple roots has either only non-negative or only non-positive coefficients. It leads (cf. [11, A.1.1 (iv)]) to the following formula for $l(w)$. Set $S(w) := \{\alpha \in \Delta_{\text{re}}^+ | w\alpha \in \Delta_{\text{re}}^-\}$. Then

$$l(w) = \text{card}\, S(w).$$

2.1.5 Given $\gamma \in \Delta_{\text{re}}^+$ one may write $\gamma = w\alpha_i$, for some $w \in W$, $\alpha_i \in \pi$. After Kac [20, Sect. 5.1] ws_iw^{-1} is independent of the choice of the pair w, α_i and one sets $s_\gamma = ws_{\alpha_i}w^{-1}$. Given $\gamma \in \Delta_{\text{re}}^+$, $w \in W$ one has $l(s_\gamma w) > l(w)$ if and only if $w^{-1}\gamma \in \Delta_{\text{re}}^+$. If $l(w_2) = 1 + l(w_1)$ and $w_2 = s_\gamma w_1$, we write $w_1 \xrightarrow{\gamma} w_2$. Note that γ is uniquely determined by the pair w_1, w_2 (if it exists) and can be omitted. Write $w < w'$, if there exists a sequence (called a Bruhat sequence) $w = w_1 \to w_2 \to \cdots \to w_m = w'$. It is an order relation on W called the Bruhat order.

Bruhat sequences play a major role in the Littelmann path model and as a consequence in describing the Kashiwara crystal $B(\infty)$—see 2.4.

2.2 *Crystals*

2.2.1 Set $P = \{\lambda \in \mathfrak{h}^* | \alpha^\vee(\lambda) \in \mathbb{Z},\ \forall \alpha^\vee \in \pi^\vee\}$ (resp. $P^+ = \{\lambda \in P | \alpha^\vee(\lambda) \geq 0,\ \forall \alpha^\vee \in \pi^\vee\}$) called the set of integral (resp. integral and dominant) weights. Observe that $Q := \mathbb{Z}\pi \subset P$.

2.2.2 A crystal B is a set with maps

(i) $\text{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$, $\forall i \in I$,
(ii) $e_i, f_i : B \cup \{0\} \to B \cup \{0\}$, with $e_i 0 = f_i 0 = 0$, $\forall i \in I$,

satisfying

(C1) For all $b \in B$, $i \in I$ one has $\varphi_i(b) = \varepsilon_i(b) + \alpha_i^\vee(\text{wt}\, b)$,
(C2) For all $b, e_i b \in B$, $i \in I$ one has $\text{wt}(e_i b) = \text{wt}\, b + \alpha_i$, $\varepsilon_i(e_i b) = \varepsilon_i(b) - 1$,

(C3) For all $b, b' \in B$, $i \in I$, one has $b' = e_i b \Leftrightarrow b = f_i b'$,
(C4) For all $b \in B$, $i \in I$ with $\varphi_i(b) = -\infty$, one has $e_i b = f_i b = 0$.

These rules need modifying in the Borcherds case [19].

2.2.3 One may view a crystal as a graph with vertices $b \in B$ and directed edges labelled by the elements of I such that if one suppresses all edges except those corresponding to fixed $i \in I$, then the graph decomposes into a disjoint union of linear graphs.

A crystal morphism is a map $\psi : B \cup \{0\} \to B' \cup \{0\}$ satisfying the following properties, for all $i \in I$.

(1) $\psi(0) = 0$,
(2) For all $b \in B$ with $\psi(b) \neq 0$, $i \in I$ one has $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$, $\operatorname{wt}\psi(b) = \operatorname{wt} b$,
(3) For all $b \in B$, $i \in I$ with $\psi(b) \neq 0$ and $\psi(e_i b) \neq 0$, one has $\psi(e_i b) = e_i \psi(b)$,
(4) For all $b \in B$, $i \in I$ with $\psi(b) \neq 0$ and $\psi(f_i b) \neq 0$, one has $\psi(f_i b) = f_i \psi(b)$.

Notice that say (3) permits $e_i \psi(b) \neq 0$ even if $e_i b = 0$. One calls B a subcrystal of B' if ψ is an embedding. In this case as a graph B is obtained from B' by deleting the vertices in $B' \setminus B$ and the edges joining them to vertices in B'. An embedding is called strict if it commutes with the $e_i, f_i : i \in I$, that is as a graph B is a component of B'.

2.2.4 Crystals are fairly arbitrary and in particular the Cartan matrix A is not invoked except in relating φ_i, ε_i which is just a matter of definitions. However, following Kashiwara we define a crystal to be upper (resp. lower) normal if $\varepsilon_i(b) = \max\{n | e_i^n b \neq 0\}$ (resp. $\varphi_i(b) = \max\{n | f_i^n b \neq 0\}$), for all $i \in I$. If both hold, a crystal is called normal. Already normality implies that $\alpha_i^\vee(\alpha_i) = 2$, $\forall i \in I$.

Again let B be a crystal and set $B_\varpi = \{b \in B | \operatorname{wt} b = \varpi\}$. If $\operatorname{card} B_\varpi < \infty$, $\forall \varpi \in P$ we may define the formal character $\operatorname{ch} B$ of B by

$$\operatorname{ch} B = \sum_{\varpi \in P} (\operatorname{card} B_\varpi) e^\varpi$$

as an element of $\mathbb{Z}P$.

If B is a normal crystal then $\operatorname{ch} B$ is W invariant. Indeed suppressing all indices except $\{i\}$ reduces B to a disjoint union of linear graphs, or i-strings, each of which has a character stable under s_i. Kashiwara further defined an action of W on elements of $\mathbb{F}_A$ (for an exposition see [16, 16.9] which follows Littelmann). However this will not be needed here. As described in [16, 16.12] Littelmann used it to compute a combinatorial character formula (see 3.1.2) for elements of $\mathbb{F}_A$.

2.2.5 Let $\mathscr{E}$ (resp. $\mathscr{F}$) denote the monoid generated by the e_i (resp. f_i): $i \in I$. A crystal $B(\lambda)$ is said to be of highest weight $\lambda \in P$ if there exists $b_\lambda \in B(\lambda)_\lambda$ such that

(1) $e_i b_\lambda = 0, \ \forall i \in I$,
(2) $\mathscr{F} b_\lambda = B(\lambda)$.

Although this seems completely analogous to the notion of a highest weight module, it is in fact a much weaker condition. For example, any crystal B admits a highest weight subcrystal $B(\lambda)$ if $B_\lambda \neq \emptyset$. Indeed, choose $b_\lambda \in B_\lambda$ and suppress all vertices not in $\mathscr{F} b_\lambda$ and all edges joining $B \setminus \mathscr{F} B_\lambda$ to $\mathscr{F} B_\lambda$.

However, already requiring a highest weight crystal $B(\lambda)$ to be normal forces $\lambda \in P^+$. More surprisingly we have the following rather easy result [16, 11.15] involving the particular form that the Cartan matrix may take.

Lemma *Suppose there exists a highest weight normal crystal for every* $\lambda \in P^+$. *Then* $\alpha_i^\vee(\alpha_j) \leq 0$, *for* i, j *distinct and* $\alpha_i^\vee(\alpha_j) \neq 0 \Leftrightarrow \alpha_j^\vee(\alpha_i) \neq 0$.

2.2.6 We sketch how to construct for each $\lambda \in P^+$, a normal highest weight crystal $B(\lambda)$ which at least for A symmetrizable has the same character as the integrable simple module $V(\lambda)$ with highest weight $\lambda \in P^+$. However without supplementary conditions there may be normal highest weight crystals which do not have this property. For example take $\pi = \{\alpha_1, \alpha_2\}$ of type A_2. Then there exists a normal highest weight crystal $B(\alpha_1 + \alpha_2)$ with card $B(\alpha_1 + \alpha_2)_0 = 1$. However the simple module $V(\alpha_1 + \alpha_2)$ of this highest weight satisfies $\dim V(\alpha_1 + \alpha_2)_0 = 2$.

2.3 Tensor Product Structure

2.3.1 Let B_2, B_1 be crystals. Kashiwara gave the Cartesian product $B_2 \times B_1$ a crystal structure. The resulting crystal is denoted as $B_2 \otimes B_1$. The tensor product is associative but not commutative.

Given $b = b_2 \otimes b_1 \in B_2 \otimes B_1$, define $\mathrm{wt}\, b = \mathrm{wt}\, b_2 + \mathrm{wt}\, b_1$,

$$\varepsilon_i(b) = \max\{\varepsilon_i(b_2),\ \varepsilon_i(b_1) - \alpha_i^\vee(\mathrm{wt}\, b_2)\} = \max\{\varphi_i(b_2), \varepsilon_i(b_1)\} - \alpha_i^\vee(\mathrm{wt}\, b_2),$$

with $\varphi_i(b)$ given by (C1), that is

$$\varphi_i(b) = \max\{\varphi_i(b_2), \varepsilon_i(b_1)\} + \alpha_i^\vee(\mathrm{wt}\, b_1).$$

Finally the action of the $e_i, f_i : i \in I$, is defined by

$$e_i(b_2 \otimes b_1) = \begin{cases} e_i b_2 \otimes b_1, & \text{if } \varphi_i(b_2) \geq \varepsilon_i(b_1), \\ b_2 \otimes e_i b_1, & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1). \end{cases}$$

$$f_i(b_2 \otimes b_1) = \begin{cases} f_i b_2 \otimes b_1, & \text{if } \varphi_i(b_2) > \varepsilon_i(b_1), \\ b_2 \otimes f_i b_1, & \text{if } \varphi_i(b_2) \leq \varepsilon_i(b_1). \end{cases}$$

Remark The difference between the rule for e_i and for f_i can be remembered by noting that the former prefers and eventually goes into the left hand factor whilst the opposite is true for the latter. This leads to an important principle which will be exploited several times—see 2.5.11, 2.5.18, 3.5.4.

2.3.2 Now let us pass to the tensor product of finitely many crystals $B_i : i = 1, 2, \dots, n$.

Given $b = b_n \otimes b_{n-1} \otimes \cdots \otimes b_1 \in B_n \otimes B_{n-1} \otimes \cdots \otimes B_1$, $i \in I$, define the Kashiwara functions $b \mapsto r_i^k(b)$ through

$$r_i^k(b) = \varepsilon_i(b_k) - \sum_{j>k} \alpha_i^\vee(\mathrm{wt}\, b_j). \qquad (*)$$

Define $\mathrm{wt}\, b = \sum_{k=1}^n \mathrm{wt}\, b_k$, $\varepsilon_i(b) = \max_k\{r_i^k(b)\}$. Define $\varphi_i(b)$ through (C1).

To define $e_i, f_i : i \in I$ on the Cartesian product, let $s_i(b)$ (resp. $l_i(b)$) to be the smallest (resp. largest) value of k such that $r_i^k(b) = \varepsilon_i(b)$. Then

(1) $e_i(b_n \otimes b_{n-1} \otimes \cdots \otimes b_1) = b_n \otimes \cdots \otimes e_i b_l \otimes \cdots \otimes b_1$, where $l = l_i(b)$,
(2) $f_i(b_n \otimes b_{n-1} \otimes \cdots \otimes b_1) = b_n \otimes \cdots \otimes f_i b_s \otimes \cdots \otimes b_1$, where $s = s_i(b)$.

This can be expressed as saying that e_i (resp. f_i) goes in at the $l_i(b)^{\mathrm{th}}$ (resp. $s_i(b)^{\mathrm{th}}$) place.

One must check that this makes the tensor product a crystal. Set $s = s_j(b)$ and suppose $f_j b_s \neq 0$. Then $f_j b \neq 0$, and

$$r_i^k(f_j b) = \begin{cases} r_i^k(b), & \text{if } k > s, \\ r_i^k(b) + 1, & \text{if } k = s, \\ r_i^k(b) + \alpha_i^\vee(\alpha_j), & \text{if } k < s. \end{cases} \qquad (*)$$

Take $i = j$ in the above. Here the definition of s forces $r_i^k(b) < r_i^s(b)$ for $k < s$. Thus the condition $e_i f_i b = b$ forces $r_i^s(b) + 1 \geq r_i^k(b) + \alpha_i^\vee(\alpha_i)$, for $k < s$. Normality has already imposed that $\alpha_i^\vee(\alpha_i) = 2$, thus we require that $r_i^k(b) \leq r_i^s(b) - 1$, for $k < s$. This follows from the previous strict inequality as long as r_i^k takes integer values. In view of $(*)$ this forces the Cartan matrix to have integer entries. Given this one may further check that the tensor product does satisfy the crystal rules and is associative.

One may conclude from 2.2.5 and the above that enough normal crystals together with the above rule for the tensor product force A to be a Cartan matrix of a Kac–Moody algebra; but not necessarily symmetizable. Thus in the Borcherds case the tensor product rule needs to be modified [18], for $i \in I_{\mathrm{im}}$, so that a tensor product of crystals is again a crystal.

2.3.3 The tensor product structure on crystals has several quite astonishing consequences. First it allows one to build interesting crystals, and in particular Kashiwara's $B(\infty)$ crystal, from very simple ones. Secondly it allows one to formulate and prove a uniqueness theorem for crystals [11, 6.4.21]. Thirdly it allows one to formulate and prove Littelmann's remarkable path independence theorem [28]. Fourthly it led via Lusztig's quantum Frobenius map to the construction of standard monomial bases [29]. Fifthly it gives an elementary proof of the refined PRV conjecture [26, 3.5] which in turn extends the Chevalley restriction theorem [12], and finally it leads to a combinatorial version of a result asserting that certain tensor products of Demazure modules admit a Demazure flag [13, 15, 21]. Some of these constructions and perhaps all go over to the Borcherds case, though here significant extra efforts are needed [17].

2.3.4 We shall certainly not discuss all these results here. In general we shall avoid the details of Littelmann theory which although very beautiful is nevertheless too much of a casse-tête chinois to inflict on an amiable audience. What we do mention is that Kashiwara's tensor product which arose from a suitable $q \to 0$ limit, does seem rather mysterious if not obscure. However, in Littelmann's theory it is an immediate and straightforward consequence of applying the Littelmann crystal operators to concatenated paths.

2.3.5 In the next section we shall want to give a countably infinite Cartesian product $\cdots \times B_2 \times B_1$, a crystal structure. In order for the summation in 2.3.2 to make sense we need to restrict to subsets of the above Cartesian product having elements $b = \{\dots, b_2, b_1\}$ satisfying $\operatorname{wt} b_j = 0$ for all but finitely many j. In order for $\varepsilon_i(b)$ to be well-defined for each $i \in I$ we must assume that $\{\varepsilon_i(b_j)\}_{j\in\mathbb{N}}$ is bounded above. We shall also assume that for all $i \in I$ one has $e_i b_j = 0$ except for finitely many j. Then if $l_i(b)$ as defined in 2.3.2 (1) is infinite, it does not matter at what point to the far left e_i enters since we still get $e_i b = 0$. In this situation we shall say that e_i enters at an infinite place.

2.4 Elementary Crystals and $B(\infty)$

2.4.1 Fix $i \in I$. The i^{th} elementary crystal denoted B_i is the set $\{b_i(-n) : n \in \mathbb{N}\}$ satisfying $\operatorname{wt} b_i(-n) = -n\alpha_i$, $\varphi_i(b_i(-n)) = -n$, $e_i b_i(0) = 0$, $e_i b_i(-n) = b_i(-(n-1))$, $f_i b_i(-n) = b_i(-(n+1))$ when $n \geq 0$, $\varphi_j(b_i(-n)) = -\infty$, $\varepsilon_j(b_i(-n)) = -\infty$, $e_j(b_i(-n)) = 0$, $f_j(b_i(-n)) = 0$, for $j \neq i$.

Notice that $\varepsilon_i(b_i(-n)) = 2n - n = \max\{k | e_i^k b_i(-n) \neq 0\}$. Thus B_i is upper normal just with respect to i. It is not at all lower normal. For brevity we often simply write $b_i(-n)$ as $-n$.

Remark Our definition modifies that of Kashiwara [23] who takes B_i to be infinite in both directions.

2.4.2 Fix a sequence $J = \{i_1, i_2, \dots\}$ of elements of I such that every element of I appears infinitely many times. We define a crystal B_J to be the subset of the countably infinite Cartesian product $\cdots \times B_{i_2} \times B_{i_1}$, in which all but finitely many entries are zero, and in particular have weight zero. Thus if b belongs to this Cartesian product the Kashiwara function $r_i^k(b)$ reduces to a finite sum and so is well defined. Then the rules in 2.3.2 give B_J a crystal structure. The unique element in which every entry is zero is denoted by b_∞. Obviously $e_i b_\infty = 0, \forall i \in I$.

Let $B_J(\infty)$ be the subcrystal of B_J generated by b_∞. It is a strict subcrystal in the sense of 2.2.3.

Obviously $B_J(\infty)_\varpi \neq 0$ implies $\varpi \in -\mathbb{N}\pi$. It follows easily from the crystal rules that $B_J(\infty) = \mathscr{F}(B_J(\infty)^{\mathscr{E}})$, where $B_J(\infty)^{\mathscr{E}} := \{b \in B_J(\infty) | e_i b = 0, \forall i \in I\}$. Obviously $\{b_\infty\} \subset B_J(\infty)^{\mathscr{E}}$, but equality is far from obvious.

Lemma *Admit that $B_J(\infty)$ is upper normal. Then $B_J(\infty)^{\mathscr{E}} = \{b_\infty\}$. In particular, $B_J(\infty)$ is a highest weight crystal.*

Proof Under the hypothesis, $b \in B_J(\infty)^{\mathscr{E}}$ implies that $\varepsilon_i(b) = 0, \ \forall i \in I$. From the definition of ε_i and the Kashiwara function, this forces $b = b_\infty$. □

2.4.3 A further deep property of $B_J(\infty)$ is that it is independent of J as a crystal, though not as a subset of $-\mathbb{N}^{\mathbb{N}}$. We write it simply as $B(\infty)$. Finally a further remarkable result is that

$$\text{ch}\, B(\infty) = \prod_{\alpha \in \Delta} \left(1 - e^{-\alpha}\right)^{-m_\alpha},$$

where m_α denote the dimension of the root subspace corresponding to α in $\mathfrak{g}_A$. This result is due to Kashiwara [22] in the symmetrizable case. In general it is obtained by combining a combinatorial character formula of Littelmann (see [16, 16.12] for an example) with the Weyl denominator identity which was proved in the required generality by Kumar [27] and by Mathieu [32].

In these lectures we shall also discuss some further remarkable properties of $B(\infty)$.

2.4.4 The above definition of $B(\infty)$ is very straightforward but not very useful for establishing many of its properties. The goal of the next section is to give a second more complicated definition which will establish these properties. We mention that our approach is a little different to that of Kashiwara in [22]. The latter construction required the Kashiwara involution which we deduce as a consequence—see 2.5.14–2.5.25.

2.5 Closed Families of Normal Highest Weight Crystals

2.5.1 Recall 2.2.5. Let $B(\lambda)$, $B(\mu)$ be highest weight crystals. From the crystal rules it is clear that $e_i(b_\lambda \otimes b_\mu) = 0, \ \forall i \in I$. On the other hand it is not at all obvious that $\mathscr{F}(b_\lambda \otimes b_\mu)$ is a subcrystal of $B(\lambda) \otimes B(\mu)$, in other words that it is $\mathscr{E}$ stable and hence a highest weight crystal.

2.5.2 Let $\mathbb{F}_A = \{B(\lambda) | \lambda \in P^+\}$ be a family of normal highest weight crystals. We say that $\mathbb{F}_A$ is closed if for all $\lambda, \mu \in P^+$, $\mathscr{F}(b_\lambda \otimes b_\mu)$ is $\mathscr{E}$ stable and isomorphic to $B(\lambda + \mu)$. A main result is that such a family is unique up to isomorphism. Moreover from it we construct $B(\infty)$ and show that it has the properties described in Sect. 2.4.

2.5.3 A closed family in the above sense was first constructed by Kashiwara [22] in the symmetrizable case by taking a $q \to 0$ limit of integrable modules over the quantized enveloping algebra. It is elementary; but involves a rather long and very intricate induction argument. A much simpler purely combinatorial argument can be obtained from the Littelmann path model [28]. It does not require symmetrizability.

2.5.4 Let us now admit the existence of a closed family $\mathbb{F}_A$ of normal highest weight crystal $\{B(\lambda) | \lambda \in P^+\}$ and prove its uniqueness, that is to say that for all $\lambda \in P^+$, the crystal $B(\lambda)$ is uniquely determined up to isomorphism. The strategy is the following. First we give a new definition of $B(\infty)$ as a direct limit of elements of $\mathbb{F}_A$. Secondly we prove an embedding theorem which allows us to express $B(\infty)$ as $B_J(\infty)$, for any sequence J defined as in 2.4.2. Finally we recover the family $\mathbb{F}_A$ from any $B_J(\infty)$. Since $B_J(\infty)$ is canonically determined by J alone, this will prove uniqueness.

2.5.5 For each $\lambda \in P^+$, let $S(-\lambda) = \{s_{-\lambda}\}$ denote the one element crystal defined by $\operatorname{wt} s_{-\lambda} = -\lambda$ and $\varepsilon_i(s_{-\lambda}) = 0, \forall i \in I$.

Let $\mathbb{F}_A = \{B(\lambda) | \lambda \in P^+\}$ be a closed family of normal highest weight crystals. Let $\lambda, \mu \in P^+$ be dominant weights. Let $\psi_{\lambda,\lambda+\mu} : B(\lambda) \otimes S(-\lambda) \to B(\lambda) \otimes B(\mu) \otimes S(-(\lambda + \mu))$ be defined by $\psi_{\lambda,\lambda+\mu}(b \otimes s_{-\lambda}) = b \otimes b_\mu \otimes s_{-(\lambda+\mu)}$.

Lemma *The map $\psi_{\lambda,\lambda+\mu}$ is a crystal embedding commuting with $\mathscr{E}$. Moreover* $\operatorname{Im} \psi_{\lambda,\lambda+\mu} \subset B(\lambda + \mu) \otimes S(-(\lambda + \mu))$.

Proof Let $\lambda, \mu \in P^+$ be dominant weights. Clearly $\psi_{\lambda,\lambda+\mu}(b \otimes s_{-\lambda}) \neq 0$ for $b \otimes s_{-\lambda} \in B(\lambda) \otimes S(-\lambda)$. One has $\varepsilon_i(b_\mu) = \varepsilon_i(s_{-\lambda}) = 0$, whilst $\varphi_i(b) \geq 0$ for a normal crystal. For all $i \in I$, one has by the tensor product rule given in 2.3.2 that

$$\varepsilon_i(b \otimes s_{-\lambda}) = \max\left\{\varepsilon_i(b), -\alpha_i^\vee(\operatorname{wt} b)\right\}$$

and

$$\varepsilon_i(b \otimes b_\mu \otimes s_{-(\lambda+\mu)}) = \max\left\{\varepsilon_i(b), -\alpha_i^\vee(\operatorname{wt} b), -\alpha_i^\vee(\operatorname{wt} b) - \alpha_i^\vee(\mu)\right\}.$$

Now for all $b \in B(\lambda)$ one has $-\alpha_i^\vee(\operatorname{wt} b) - \alpha_i^\vee(\mu) \leq -\alpha_i^\vee(\operatorname{wt} b)$, since μ is dominant, and one has that $-\alpha_i^\vee(\operatorname{wt} b) = \varepsilon_i(b) - \varphi_i(b) \leq \varepsilon_i(b)$, by (C1) and normality. Thus

$$\varepsilon_i\left(\psi_{\lambda,\lambda+\mu}(b \otimes s_{-\lambda})\right) = \varepsilon_i(b \otimes b_\mu \otimes s_{-(\lambda+\mu)}) = \varepsilon_i(b) = \varepsilon_i(b \otimes s_{-\lambda}). \qquad (*)$$

Hence $\psi_{\lambda,\lambda+\mu}$ commutes with ε_i for all $i \in I$. Obviously

$$\operatorname{wt} \psi_{\lambda,\lambda+\mu}(b \otimes s_{-\lambda}) = \operatorname{wt} b - \lambda = \operatorname{wt}(b \otimes s_{-\lambda})$$

for all $i \in I$. Thus by (C1) and the above, $\psi_{\lambda,\lambda+\mu}$ commutes with φ_i for all $i \in I$.

The tensor product formulae given in 2.3.2, implies that for all $b \otimes s_{-\lambda} \in B(\lambda) \otimes S(-\lambda)$ and $i \in I$ one has

$$e_i \psi_{\lambda,\lambda+\mu}(b \otimes s_{-\lambda}) = e_i(b \otimes b_\mu \otimes s_{-(\lambda+\mu)}) = e_i b \otimes b_\mu \otimes s_{-(\lambda+\mu)}$$
$$= \psi_{\lambda,\lambda+\mu}\big(e_i(b \otimes s_{-(\lambda+\mu)})\big).$$

Thus $\psi_{\lambda,\lambda+\mu}$ commutes with $\mathscr{E}$. Similarly, commutation with f_i only fails when $\varphi_i(b) = 0$. For a normal crystal, this is equivalent to $f_i b = 0$.

Therefore $\psi_{\lambda,\lambda+\mu}$ is a crystal embedding of $B(\lambda) \otimes S(-\lambda)$ into $B(\lambda) \otimes B(\mu) \otimes S(-(\lambda+\mu))$ and it commutes with $\mathscr{E}$.

Now for the second claim of the lemma observe that one has

$$\begin{aligned}\psi_{\lambda,\lambda+\mu}\big(B(\lambda) \otimes S(-\lambda)\big) &= \psi_{\lambda,\lambda+\mu}\big(\mathscr{F}(b_\lambda \otimes s_{-\lambda})\big) \subset \mathscr{F}\psi_{\lambda,\lambda+\mu}(b_\lambda \otimes s_{-\lambda})\\ &= \mathscr{F}(b_\lambda \otimes b_\mu \otimes s_{-(\lambda+\mu)}) = \mathscr{F}(b_\lambda \otimes b_\mu) \otimes S\big(-(\lambda+\mu)\big)\\ &= B(\lambda+\mu) \otimes S\big(-(\lambda+\mu)\big),\end{aligned}$$

where the penultimate step follows from the normality of $B(\mu)$ as above, and the last step results from $\mathbb{F}_A$ being a closed family. □

Remark The reader should be aware that $s_{-\lambda} \otimes s_{-\mu}$ does not quite identify with $s_{-\lambda-\mu}$ since $\varepsilon_i(s_{-\lambda-\mu}) = 0$ while $\varepsilon_i(s_{-\lambda} \otimes s_{-\mu}) = \alpha_i^\vee(\lambda)$, which is in general nonzero.

2.5.6 View P^+ as a directed set through the order relation $\lambda \succcurlyeq \mu$ given $\lambda - \mu \in P^+$. Through the embeddings defined in 2.5.5 we may form the set theoretic direct limit

$$B(\infty) = \varinjlim \big(B(\lambda) \otimes S(-\lambda)\big).$$

We give $B(\infty)$ the structure of crystal as follows. Take $b \in B(\infty)$. Then $b \in B(\lambda) \otimes S(-\lambda)$ for some $\lambda \in P^+$, which is assumed to be sufficiently large so that the actions of the crystal operations (particularly $\mathscr{F}$) do not depend on λ. From Lemma 2.5.5, we obtain the following

Proposition *The above construction endows $B(\infty)$ with the structure of an upper normal highest weight crystal of highest weight* 0.

2.5.7 Recall the elementary crystals $B_i : i \in I$ defined in 2.4.1. The following result is due to Kashiwara in the symmetrizable case [22]. However the present proof is rather different and valid in general. It relies on the existence of the family $\mathbb{F}_A$ of 2.5.5.

Theorem *Fix $i \in I$.*

(i) *There exists a unique crystal embedding*

$$\psi_i : B(\infty) \hookrightarrow B(\infty) \otimes B_i,$$

sending b_∞ to $b_\infty \otimes b_i(0)$.

(ii) *If* $b \otimes b_i(-n) \in \operatorname{Im} \psi_i$, *then* $\varphi_i(b) \geq 0$.
(iii) *The crystal embedding* ψ_i *is strict.*

Proof (i) Uniqueness is obvious since b_∞ is a generator of $B(\infty)$.

Take $\lambda, \mu \in P^+$. By the closure property of $\mathbb{F}_A$, the elements $b_\lambda \otimes b_\mu$ and $b_{\lambda+\mu}$ generate the same highest weight crystal, namely $B(\lambda + \mu)$.

Let $b \in B(\infty)$. Then $b = fb_\infty$ for some $f \in \mathscr{F}$. Choose $\gamma \in P^+$ such that $\alpha_j^\vee(\gamma) : j \in I$ are positive and are sufficiently large relative to f. Write $\gamma = \lambda + \mu$ for $\lambda, \mu \in P^+$ such that $\alpha_i^\vee(\lambda) = 0$ and $\alpha_j^\vee(\mu) = 0,\ \forall j \in I \setminus \{i\}$. Then $fb_\infty = fb_{\lambda+\mu} = f(b_\lambda \otimes b_\mu) = f'b_\lambda \otimes f_i^m b_\mu$ for some $f' \in \mathscr{F}$ and $m \in \mathbb{N}$. Define $\psi_i(b) := f'b_\infty \otimes b_i(-m)$.

First we show that ψ_i is well-defined and commutes with $\mathscr{F}$. In particular, we must show that if applying f to $b_\infty \otimes b_i(0)$ gives $f'b_\infty \otimes b_i(-m)$, for some $f' \in \mathscr{F}$, then applying f to $b_\lambda \otimes b_\mu$ gives $f'b_\lambda \otimes f_i^m b_\mu$, and the latter is non-zero. Assume the claim holds for some $f \in \mathscr{F}$ and verify it for $f_j f : j \in I$.

Suppose $j \in I \setminus \{i\}$. Then we must show that f_j enters in the left hand factor in both cases, that is for $B(\infty) \otimes B_i$ and for $B(\lambda) \otimes B(\mu)$. This is trivial in the first case. For the second case we must show that $\varphi_j(f'b_\lambda) > \varepsilon_j(f_i^m b_\mu)$. Now $e_j f_i^m b_\mu = 0$, since otherwise it is a non-zero element of $B(\mu)$ of weight $\mu - m\alpha_i + \alpha_j$. Since $\alpha_i \neq \alpha_j$, this does not lie in $\mu - \mathbb{N}\pi$ and hence not in the set of weights of $B(\mu)$. By the upper normality of $B(\mu)$ one obtains $\varepsilon_j(f_i^m b_\mu) = 0$. Now by (C1) one has

$$\varphi_j(f'b_\lambda) = \varepsilon_j(f'b_\lambda) + \alpha_j^\vee(f'b_\lambda) \geq \alpha_j^\vee(f'b_\lambda) = \alpha_j^\vee(\operatorname{wt} f' + \lambda),$$

since $\varepsilon_j(f'b_\lambda) \geq 0$, by the upper normality of $B(\lambda)$. Thus it suffices that $\alpha_j^\vee(\lambda) > -\alpha_j^\vee(\operatorname{wt} f')$. Moreover the resulting element of $B(\lambda) \otimes B(\mu)$ is non-zero.

Finally suppose that $j = i$. Since $\alpha_i^\vee(\lambda) = 0$, it follows from (C1) and 2.5.5 $(*)$ that $\varphi_i(f'b_\lambda) = \varphi_i(f'b_\infty)$, which is independent of λ. Again $\varepsilon_i(b_i(-m)) = m = \varepsilon_i(f_i^m b_\mu)$, by the upper normality of $B(\mu)$. Then through the tensor product rules it follows that f_i enters in the same factor for both cases. Moreover the resulting element of $B(\lambda) \otimes B(\mu)$ is non-zero given that $\alpha_i^\vee(\mu) \geq m + 1$. This proves (i).

(ii) now follows from the normality of $B(\lambda)$, which forces $\varphi_i(b) \geq 0$, for $b \in B(\lambda)$ and that viewed as an element of $B(\infty)$ the value of $\varphi_i(b)$ is decreased by $\alpha_i^\vee(\lambda) = 0$.

For (iii) one must show that all the crystal operators commute with the crystal maps. This was already shown in the proof of (i) for the elements of $\mathscr{F}$ which are also shown to act injectively. Again since ε, φ, wt commute with ψ_i on the generators, they must also commute on all elements.

It remains to consider the $e_j : j \in I$. Clearly $e_j\psi_i(b) = \psi_i(e_j b)$ if $b = f_j b'$, equivalently if $e_j b \neq 0$. Thus we need only show that $e_j b = 0$ implies $e_j\psi_i(b) = 0$. Since $B(\infty)$ is upper normal, it is enough to show that $\varepsilon_j(\psi_i(b)) = 0$ implies $e_j\psi_i(b) = 0$. Let we write $\psi(b) = b' \otimes b_i(-n)$. Then since $\varepsilon_j(b') \geq 0$, by the upper normality of $B(\infty)$, the condition $\varepsilon_j(\psi_i(b)) = 0$, forces e_j to enter the left hand factor with $\varepsilon_j(b') = 0$. This forces $e_j b' = 0$, again by upper normality and so $e_j\psi_i(b) = 0$, as required. □

2.5.8 From Theorem 2.5.7, it follows that for all $n \in \mathbb{N}^+$ and all $i_1, i_2, \dots, i_n \in I$, there exists a unique strict embedding $B(\infty) \hookrightarrow B(\infty) \otimes B_{i_n} \otimes \cdots \otimes B_{i_1}$, sending b_∞ to $b_\infty \otimes (b_{i_n}(0) \otimes \cdots \otimes b_{i_1}(0))$. If $f \in \mathscr{F}$ applied to such an expression goes into the right hand factor, then for all $j \in I$ and $m > n$ for which $i_m = j$, the tensor product rule implies that $f_j f$ goes into the right hand factor of $b_\infty \otimes (b_{i_m}(0) \otimes \cdots \otimes b_{i_1}(0))$. Now take J as in 2.4.2. Then we obtain a unique strict embedding of $B(\infty)$ into $B(\infty) \otimes B_J$ sending b_∞ to $b_\infty \otimes (\cdots \otimes b_{i_m}(0) \otimes \cdots \otimes b_{i_1}(0))$. From this it is clear that $B_J(\infty)$ is isomorphic to $B(\infty)$ as defined in 2.5.6, hence is upper normal and independent of J.

2.5.9 We now recover $\mathbb{F}_A$ from $B(\infty)$ as defined in 2.5.6.

Fix $\lambda \in P^+$ and $S(\lambda) = \{s_\lambda\}$ denote the one-element crystal defined by $\mathrm{wt}\, s_\lambda = \lambda$ and $\varphi_i(s_\lambda) = 0,\ \forall i \in I$. Note the subtle difference with the definition of $S(-\lambda)$ given in 2.5.5.

Lemma *$\mathscr{F}(b_\infty \otimes s_\lambda)$ is a strict subcrystal of $B(\infty) \otimes S(\lambda)$ and is isomorphic to the crystal $B(\lambda)$ of the family $\mathbb{F}_A$.*

Proof Since

$$\varepsilon_i(s_\lambda) = \varphi_i(s_\lambda) - \alpha_i^\vee(\lambda) = -\alpha_i^\vee(\lambda),$$

the tensor product rules give

$$f_i(b \otimes s_\lambda) = f_i b \otimes s_\lambda \quad \Leftrightarrow \quad \varphi_i(b) + \alpha_i^\vee(\lambda) > 0.$$

On the other hand by the lower normality of $B(\lambda) \in \mathbb{F}_A$, and in view of the shift by λ in the value of $\mathrm{wt}\, b$, the image of $b \in B(\lambda)$ in $B(\infty)$ satisfies $f_i b = 0$, if and only if $\varphi_i(b) + \alpha_i^\vee(\lambda) \le 0$.

Again since $\varphi_i(s_\lambda) = 0$, the upper normality of $B(\infty)$ ensures that the $e_i : i \in I$ always enter the first factor and that the $\varepsilon_i : i \in I$ are preserved. Obviously wt is preserved and hence so are the φ_i.

This proves the first assertion.

The last assertion follows easily from the presentation of $B(\infty)$ given in 2.5.6 and the above observations, the essential point being that the crystal operators $e_i, f_i : i \in I$ enter the first factor in both cases with the slight exception noted for the $f_i : i \in I$. □

Remark Again as in Remark 2.5.10 one cannot identify $s_\lambda \otimes s_\mu$ with $s_{\lambda+\mu}$. However for $\lambda \in P^+$ we *can* identify $s_{-\lambda} \otimes s_\lambda$ with s_0. This again proves that the embedding $b \mapsto b \otimes s_{-\lambda}$ of $B(\lambda)$ into $B(\infty)$ composed with $-\otimes s_\lambda$ on its image is a crystal automorphism of $B(\lambda)$.

2.5.10

Corollary *The closed family $\mathbb{F}_A$ of normal highest weight crystals is unique up to crystal isomorphism.*

Proof Lemma 2.5.9 recovers $B(\lambda)$ from $B(\infty)$ and hence by 2.5.8 from $B_J(\infty)$. Yet $B_J(\infty)$ is defined (2.4.2) in a manner independent of $\mathbb{F}_A$. Thus any given member of the family $B(\lambda)$ is uniquely determined by $\lambda \in P^+$. □

Remark If we start from $B_J(\infty)$ constructed as in 2.4.2 and admit that it is upper normal, then from Lemma 2.4.2 and pursuing the reasoning in Lemma 2.5.9 it follows easily that $B(\lambda) := \mathscr{F}(b_\infty \otimes s_\lambda)$ is a normal crystal of highest weight λ. On the other hand it is not so obvious that the resulting family $\{B(\lambda) : \lambda \in P^+\}$ is closed.

2.5.11 Using upper normality we can obtain a more precise version of 2.5.7.

If B' is a subset of a crystal which is $\mathscr{E}$ stable, then it makes sense to ask if B' is upper normal. Let $\mathscr{E}^i$ (resp. $\mathscr{F}^i$) denote the monoid generated by the e_j (resp. f_j) : $j \in I - \{i\}$. Define ψ_i by the conclusion of 2.5.7. For each $i \in I$, let B^i denote the subset of $B(\infty)$ defined by

$$B^i = \left\{b \in B(\infty) | b \otimes b_i(-n) \in \operatorname{Im} \psi_i, \text{ for some } n \in \mathbb{N}\right\}.$$

Define $B' = B^i \otimes B_i \subset B(\infty) \otimes B_i$. Clearly $\operatorname{Im} \psi_i \subset B'$.

Lemma *Fix $i \in I$.*

(i) *B^i is $\mathscr{E}$ stable and upper normal,*
(ii) *B^i is $\mathscr{F}^i$ stable,*
(iii) *$B(\infty) = B^i \times B_i$ as a set.*

Proof Obviously $f_j B^i \subset B^i$ and $f_j B' \subset B'$, if $j \in I \setminus \{i\}$. This proves (ii).

Again $e_j B^i \subset B^i$ and $e_j B' \subset B'$, if $j \in I \setminus \{i\}$. Let us show this also hold for i. Take $b \in B^i$ and $n \in \mathbb{N}$ such that $b \otimes b_i(-n) \in \operatorname{Im} \psi_i$. We claim that after sufficiently many applications e_i enters into the left hand factor. Otherwise $b \otimes b_i(0) \in \operatorname{Im} \psi_i$. Yet $\varepsilon_i(b_i(0)) = 0 \le \varphi_i(b)$, by (ii) of Theorem 2.5.7. Then $e_i(b \otimes b_i(0)) = e_i b \otimes b_i(0) \in \operatorname{Im} \psi_i$ and so $e_i b \in B^i$, as required.

We conclude that B^i and hence B' is $\mathscr{E}$ stable. Thus it makes sense to consider if B' is upper normal. Observe that $b \otimes b_i(-m) \in \operatorname{Im} \psi_i$, means that $b \in B(\infty)$. Thus the upper normality of $B(\infty)$ implies that B' is upper normal with respect to all indices except possibly i.

Let us show that B' is i-upper normal. Observe that the value of r_i^k on $\{b \otimes b_i(-n) : n \in \mathbb{N}\}$ is independent of n for $k > 1$. Suppose $b \otimes b_i(-n) \in \operatorname{Im} \psi_i$ and set $t = r_i^1(b \otimes b_i(-n)) - \max_{k>1} r_i^k(b \otimes b_i(-n)) = n - \alpha_i^\vee(\operatorname{wt} b) - \varepsilon_i(b) = n - \varphi_i(b) \le n$, since $\varphi_i(b) \ge 0$, by (ii) of 2.5.7. If $t > 0$, then $e_i^t(b \otimes b_i(-n)) = b \otimes b_i(-(n-t)) \in \operatorname{Im} \psi_i$ and $r_i^1(b \otimes b_i(-(n-t)) = \max_{k>1} r_i^k(b \otimes b_i(-(n-t))$. Thus it suffices to establish upper normality of B' when $r_i^1(b \otimes b_i(-n)) \le \max_{k>1} r_i^k(b \otimes b_i(-n))$. In this case e_i goes into the left hand factor. Moreover the value of r_i^1 decreases by $\alpha_i^\vee(\alpha_i) = 2$, whilst the value of $\max_{k>1} r_i^k$ decreases by 1. Thus powers of e_i continue to go into the left hand factor. (This is the principle referred to in Remark 2.3.1.) We conclude that the upper normality of $B(\infty)$ implies the upper normality of B'. This proves (i).

By the upper normality of $B' \subset B_J$ it follows as in the proof of Lemma 2.4.2 that $B'^{\mathscr{E}} = \{b_\infty\}$. Yet the set of weights of B' lie in $-\mathbb{N}\pi$, so for all $b \in B'$ one has $b_\infty \in \mathscr{E}b$. Combined with our previous assertion this implies that $B' = \mathscr{F}B'^{\mathscr{E}} = \mathscr{F}b_\infty = B(\infty)$. Hence (iii). □

2.5.12 By the above lemma we can give B^i a crystal structure by taking the induced structure from $B(\infty)$, except with respect to f_i. Now given $b \in B^i$ one has $b \otimes b_i(0) \in B(\infty)$, by (iii) above. Then $f_i(b \otimes b_i(0)) = f_i b \otimes b_i(0)$ and so defines $f_i b$ except if $\varphi_i(b) = 0$. In the latter case we redefine $f_i b$ to be equal to zero.

Lemma *Fix $i \in I$.*

(i) *B^i is f_i stable and i-lower normal,*
(ii) *B^i is a subcrystal of $B(\infty)$,*
(iii) *$B(\infty) = B^i \otimes B_i$ as a crystal.*

Proof Consider $b' = b \otimes b_i(0)$, $b \in B^i$. By the tensor product rules one has $f_i b' = f_i b \otimes b_i(0)$, unless $\varphi_i(b) = \varepsilon_i(b_i(0)) = 0$. In the latter case, one has that $f_i b = 0$ by our definition. Hence B^i is f_i stable. By (ii) of Theorem 2.5.7 we obtain $\varphi_i(b) = \max\{\varphi_i(b'), 0\} = \varphi_i(b')$, whilst by construction $f_i b = 0 : b \in B^i$, if and only if $\varphi_i(b) = 0$. Thus B^i is i-lower normal. Hence (i).

The fact that B^i is a subcrystal of $B(\infty)$ now follows from (i) and Lemma 2.5.11. Moreover this also shows that $B(\infty) = B^i \otimes B_i$ as a crystal. □

2.5.13

Lemma *Fix $i \in I$. Then B^i is the unique i-normal subcrystal of $B(\infty)$ such that the crystal embedding $B^i \hookrightarrow B(\infty)$ commutes with $\mathscr{E}, \mathscr{F}^i$.*

Proof Since B^i is $\mathscr{E}$ stable, it contains b_∞. Since B^i is a subcrystal of $B(\infty) = \mathscr{F}b_\infty$, it is generated by b_∞. Since it is i-lower normal, we have for $b \in B^i$ that $\varphi_i(b) = 0$ implies $f_i b = 0$. Clearly B^i is the unique subset with this property which is f_i-stable and commutes with $\mathscr{E}, \mathscr{F}^i$. □

2.5.14 In view of Lemma 2.5.12 we may define a new pair of operators $e_i^\star, f_i^\star$ on $B(\infty) = B^i \otimes B_i$ as just e_i, f_i acting on the right hand factor. Then

$$B(\infty)^{e_i^\star} = B^i = \left\{b \otimes b_i(0) \in \operatorname{Im} \psi_i\right\}.$$

Since $\mathrm{wt}(b \otimes b_i(-m)) = \mathrm{wt}\, b - m\alpha_i$, it follows that the first part of (C2) holds.

Further set $\mathrm{wt}^\star = \mathrm{wt}$ and $\varepsilon_j^\star(b) = \max\{m \in \mathbb{N}^+ | e_j^{\star m} b \neq 0\}$. Observe that $\varepsilon_i^\star(b) = n$ for $b = b' \otimes b_i(-n)$. However we do not know how to calculate $\varepsilon_j^\star(b) : j \in I \setminus \{i\}$ in this presentation of $B(\infty)$. The importance of this question is discussed in 3.2.6.

Define $\varphi_i^\star(b)$ through (C1).

By construction $B(\infty)$ with the maps $e_i^\star, f_i^\star, \varepsilon_i^\star, \varphi_i^\star, \mathrm{wt}^\star := \mathrm{wt}$ is an upper normal crystal. However since we have no presentation of $B(\infty)^\star$ in which we can simultaneously calculate the values of the $\varepsilon_j^\star : j \in I$ we cannot immediately conclude as in 2.4.2 that $B(\infty)^{\mathscr{E}^\star} = \{b_\infty\}$.

When we want to emphasize the $\star$ crystal structure of $B(\infty)$ we shall write it as $\mathrm{s}B(\infty)^\star$.

Let $\mathscr{E}^\star$ (resp. $\mathscr{F}^\star$) denote the monoid generated by the $e_j^\star$ (resp. $f_j^\star$) : $j \in I$.

2.5.15 We may observe (as did Kashiwara in the symmetrizable case) that the above crystal structures are (almost!) independent. This is expressed by the

Lemma *Take $i, j \in I$ distinct. Then the pairs $e_i, f_j^\star$; $f_i, f_j^\star$; $e_i^\star, f_j$; $e_i^\star, f_j$ commute.*

Proof Consider $e_i, f_j^\star$. Identify $B(\infty)$ with its image under the embedding ψ_j. Write $b \in B(\infty)$ in the form $f_j^{\star m} b'$ with $e_j^\star b' = 0$. Thus $e_i f_j^\star b = e_i f_j^{\star(m+1)} b' = e_i(b' \otimes b_j(-(m+1)) = e_i b' \otimes b_j(-(m+1))$. We conclude that $e_i b' \in B(\infty)^{e_j^\star}$ and this last expression equals $f_j^\star e_i b$. Since $b \in B(\infty)$ is arbitrary it follows that $e_i, f_j^\star$ commute on $B(\infty)$. The remaining cases are similar. □

2.5.16 By contrast to 2.5.15, the $e_i, e_i^\star$ do not commute. On the other hand we showed in the proof of 2.5.11 that B^i which now identifies with $B(\infty)^{e_i^\star}$ is $\mathscr{E}$ stable and upper normal. Hence $B(\infty)^{\mathscr{E}^\star} = \bigcap_{i \in I} B(\infty)^{e_i^\star}$ is $\mathscr{E}$ stable.

Lemma $B(\infty)^{\mathscr{E}^\star} = \{b_\infty\}$, $B(\infty) = \mathscr{F}^\star b_\infty$.

Proof Obviously $B(\infty)^{\mathscr{E}^\star} \supset \{b_\infty\}$. Suppose b belongs to the complement. Since $B(\infty)^{\mathscr{E}^\star}$ is $\mathscr{E}$ stable and $B(\infty)^{\mathscr{E}} = \{b_\infty\}$ we can find $e \in \mathscr{E}$ such that $eb = b_\infty$. Choose $i \in I$ such that $e = e_i e'$, for some $e' \in \mathscr{E}$. Then $b' := e'b \in B(\infty)^{\mathscr{E}^\star}$. Yet $e_i b' = b_\infty$, so $b = f_i b_\infty = f_i(b_\infty \otimes b_i(0)) = b_\infty \otimes b_i(-1)$, by the tensor product rules. Yet obviously $b_\infty \otimes b_i(-1) \notin B(\infty)^{\mathscr{E}^\star}$. Hence a contradiction. This proves the first part. The second part is a consequence of the first part. □

Hence $B(\infty)$ with the maps $e_i^\star, f_i^\star, \varepsilon_i^\star, \varphi_i^\star, \mathrm{wt}^\star := \mathrm{wt}$ is a highest weight crystal of highest weight zero.

2.5.17 Let $\mathscr{E}^{\star i}$ (resp. $\mathscr{F}^{\star i}$) denote the monoid generated by the $e_j^\star$ (resp. $f_j^\star$) : $j \in I - \{i\}$. Then $B(\infty)^{e_i}$ is $\mathscr{E}^{\star i}$ and $\mathscr{F}^{\star i}$ stable by 2.5.15. It remains to consider the action of $e_i^\star$ and $f_i^\star$.

Lemma *Take $b \in B(\infty)^{e_i}$. Then*

(i) $e_i^\star b \in B(\infty)^{e_i}$,
(ii) $\varphi_i^\star(b) \geq 0$,
(iii) $f_i^\star b \in B(\infty)^{e_i} \Leftrightarrow \varphi_i^\star(b) > 0$.

Proof We use the presentation $B(\infty) = B^i \otimes B_i$. Take $b = b' \otimes b_i(-m) \in B(\infty)^{e_i}$ and recall that $\varphi_i(b') \geq 0$, by (ii) of Theorem 2.5.7. Then $e_i b = 0$, if and only if $e_i b = e_i b' \otimes b_i(-m)$ and $e_i b' = 0$. In particular $\varphi_i(b') \geq \varepsilon_i(b_i(-m)) = m$.

Now $e_i^\star b = b' \otimes b_i(-(m-1))$ and $\varphi_i(b') \geq m > m-1 = \varepsilon_i(b_i(-(m-1)))$. Hence $e_i e_i^\star b = e_i b' \otimes b_i(-(m-1)) = 0$. This proves (i).

For (ii) recall that $\varepsilon_i^\star(b' \otimes b_i(-m)) = m$. Since $e_i b' = 0$, as in the first part we have $\varepsilon_i(b') = 0$, by the upper normality of $B(\infty)$ and so $\alpha_i^\vee(\mathrm{wt}\, b') = \varphi_i(b') - \varepsilon_i(b') = \varphi_i(b') \geq m$. Now $\alpha_i^\vee(\mathrm{wt}\, b) = \alpha_i^\vee(\mathrm{wt}\, b' - m\alpha_i) = \varphi_i(b') - 2m$. Hence $\varphi_i^\star(b) = \varepsilon_i^\star(b) + \alpha_i^\vee(\mathrm{wt}\, b) = m + \varphi_i(b') - 2m = \varphi_i(b') - m \geq 0$. Hence (ii).

Suppose $f_i^\star b \notin B(\infty)^{e_i}$. By definition $f_i^\star b = b' \otimes b_i(-(m+1))$ and so $e_i f_i^\star b \neq 0$, implies that $e_i(f_i^\star b) = b' \otimes e_i b_i(-(m+1)) = b' \otimes b_i(-m)$. In particular $\varphi_i(b') < \varepsilon_i(b_i(-(m+1))) = m+1$. Yet $\varphi_i(b') \geq m$, by the first part and so $\varphi_i(b') = m$, implying $\varphi_i^\star(b) = 0$. Conversely, if $f_i^\star b \in B(\infty)^{e_i}$ then $0 = e_i f_i^\star b = e_i b' \otimes b_i(-m)$ and $\varphi_i(b') \geq \varepsilon_i(b_i(-(m+1))) = m+1$, implying $\varphi_i^\star(b) > 0$. This gives (iii). □

2.5.18 We can give $B(\infty)^{e_i}$ a $\star$ crystal structure by setting $f_i^\star b$ to be equal to zero when $\varphi_i^\star(b) = 0$. Denote this crystal by $B^{i\star}$. Then $B^{i\star}$ is a subcrystal of $B(\infty)^\star$ with the rule $\varphi_i^\star(b) = 0 \Rightarrow f_i^\star b = 0$. In particular the embedding $B^{i\star} \hookrightarrow B(\infty)^\star$ commutes with $\mathscr{E}^\star$ and $\mathscr{F}^{\star i}$.

Since $B(\infty)^\star = \mathscr{F}^\star b_\infty$ by Lemma 2.5.16 we may define a map of $B(\infty)^\star$ into $B^{i\star} \otimes B_i$ by sending $f^\star b_\infty \mapsto f^\star(b_\infty \otimes b_i(0))$.

Lemma $B(\infty)^\star \xrightarrow{\sim} B^{i\star} \otimes B_i$.

Proof Take $b = b' \otimes b_i(-n) \in B^{i\star} \otimes B_i$. Obviously each element of $\mathscr{E}^{\star i}$ enters the left hand factor. By (ii) of Lemma 2.5.17 this is eventually true of $e_i^\star$, as shown in the proof (given in (ii) of Lemma 2.5.11) of the analogous result for e_i. Now suppose $b \in (B^{i\star} \otimes B_i)^{\mathscr{E}^\star}$. By the above each $e_j : j \in I$ enters the left hand factor and so $b' \in (B(\infty)^\star)^{\mathscr{E}^\star} = \{b_\infty\}$, by 2.5.16. Thus $b' = b_\infty$, which further implies that $n = 0$. As in the proof of (iii) of Lemma 2.5.11, this proves surjectivity. □

2.5.19 Thus we have a crystal embedding $B(\infty)^\star \xrightarrow{\sim} B^{i\star} \otimes B_i \hookrightarrow B(\infty)^\star \otimes B_i$. Since this holds for all $i \in I$, we conclude that $B(\infty)^\star$ also satisfies 2.5.8. This presentation implies that $B(\infty)^\star$ with the $\star$ action is isomorphic to $B(\infty)$ with the previous action. Thus we obtain

Theorem *$B(\infty)^\star$ is isomorphic to $B(\infty)$.*

2.5.20 Let us show how the above result allows us to obtain the Kashiwara involution on $B(\infty)$ in the general, not necessarily symmetrizable, case.

Given $b \in B(\infty)$, we can write $b = f_{i_1}^{m_1} f_{i_2}^{m_2} \cdots f_{i_k}^{m_k} b_\infty$. Define $b^\star$ by replacing $f_{i_j}^{m_j}$ by $f_{i_j}^{\star m_j}$. Since the $f_i^\star : i \in I$ act injectively $b^\star \neq 0$. By the isomorphism theorem above, the result is independent of choice of representatives. This defines a map $b \mapsto b^\star$ of $B(\infty)$ into $B(\infty)^\star$. Notice that we have $(fb)^\star = f^\star b^\star$, for all $f \in \mathscr{F}$,

$b \in B(\infty)$. A map inverse to $\star$ may be similarly defined by expressing b as an element of $\mathscr{F}^\star b_\infty$ and replacing $f_i^\star$ by f_i. Hence $\star$ is bijective and intertwines the two crystal structures on $B(\infty)$. Comparison with [11, 6.1.12, 6.1.13] shows that $\star$ coincides with the Kashiwara involution if A is symmetrizable.

2.5.21 We would like to show that the map $\star$ as defined above is an involution in the not necessarily symmetrizable case. First we will study the relationship between these two different crystal structures on $B(\infty)$.

Let us recall the decomposition $B(\infty)^\star \xrightarrow{\sim} B^{i\star} \otimes B_i$ given by the star action. We may identify $B(\infty)^\star$ with $B(\infty)$ as sets, and we also have the decomposition $B(\infty) \xrightarrow{\sim} B^i \otimes B_i$.

Take $b \in B(\infty)^\star$ and write $b = b' \otimes b_i(-m)$ as an element of $B^{i\star} \otimes B_i$. Now $\varepsilon_i^\star(b) = \max\{m, \varphi_i^\star(b')\} - \alpha_i^\vee(\mathrm{wt}\, b')$. Since $\alpha_i^\vee(\mathrm{wt}\, b') = 2m + \alpha_i^\vee(\mathrm{wt}\, b)$, we obtain

$$\varepsilon_i^\star(b) \geq -m - \alpha_i^\vee(\mathrm{wt}\, b), \text{ with equality} \Leftrightarrow \varphi_i^\star(b') \leq m \Leftrightarrow f_i^\star b = f_i^{\star\star} b. \qquad (*)$$

Indeed the second equivalence follows from the fact that by the definition of $f_i^{\star\star}$ it always enters the right hand factor, whilst $f_i^\star$ enters the right hand factor (and hence $f_i^\star b = f_i^{\star\star} b$) if and only if $\varphi_i^\star(b') \leq \varepsilon_i^\star(b_i(-m)) = m$.

Now also write $b = b'' \otimes b_i(-n)$ as an element of $B^i \otimes B_i$. Then as in $(*)$ we obtain

$$\varepsilon_i(b) \geq -n - \alpha_i^\vee(\mathrm{wt}\, b), \text{ with equality} \Leftrightarrow \varphi_i(b'') \leq n \Leftrightarrow f_i b = f_i^\star b. \qquad (**)$$

By the definition of $\star$ crystal structure $\varepsilon_i^\star(b) = n$.

Lemma *In the presentation $b = b' \otimes b_i(-m) \in B^{i\star} \otimes B_i$, one has $\varepsilon_i(b) = m$.*

Proof Given $\lambda \in \mathbb{N}\pi$, let $|\lambda|$ denote the sum of its coefficients. The proof is by induction on $|-\mathrm{wt}(b)|$. It is trivial when $|-\mathrm{wt}(b)| = 0$.

Recall (2.5.16) that $B(\infty)^\star = \mathscr{F}^\star b_\infty$. If $j \in I \setminus \{i\}$, then $f_j^\star$ enters the first factor and moreover by 2.5.15 and upper normality one has $\varepsilon_i(f_j^\star b) = \varepsilon_i(b)$. Thus it remains to show that the conclusion of the lemma holds for $f_i^\star b$ given that it holds for b.

Case (1). One has $f_i^\star(b' \otimes b_i(-m)) = b' \otimes b_i(-(m+1))$.

By the tensor product rule this holds if and only if $\varphi_i^\star(b') \leq m$. On the other hand $\varepsilon_i^\star(b) = n$, whilst $\varepsilon_i(b) = m$ by the induction hypothesis. Substitution into $(*)$ above gives $m + n = -\alpha_i^\vee(\mathrm{wt}\, b)$. Further substitution into $(**)$ above gives $f_i b = f_i^\star b$. Consequently $\varepsilon(f_i^\star b) = \varepsilon_i(f_i b) = \varepsilon_i(b) + 1$, as required.

Case (2). One has $f_i^\star(b' \otimes b_i(-m)) = f_i^\star b' \otimes b_i(-(m))$.

By the tensor product rule this holds if and only if $\varphi_i^\star(b') > m$. Substituting in $(*)$ and $(**)$ above and using the induction hypothesis as before gives $\varphi_i(b'') > n$. Thus $f_i^\star b = b'' \otimes b_i(-(n+1))$, and then

$$\varepsilon_i(f_i^\star b) = \max\big\{\varphi_i(b''), n+1\big\} - \alpha_i^\vee(\mathrm{wt}\, b'') = \varphi_i(b'') - \alpha_i^\vee(\mathrm{wt}\, b'') = \varepsilon_i(b),$$

as required. □

2.5.22

Corollary *Let $b \in B(\infty)$ and identify $B(\infty)$ with $B(\infty)^\star$ as sets. Then the following are equivalent*:

(i) $f_i b = f_i^\star b$,
(ii) $f_i^\star b = f_i^{\star\star} b$,
(iii) *Equality holds in* $\varepsilon_i^\star(b) \geq -\varepsilon_i(b) - \alpha_i^\vee(\mathrm{wt}\, b) = -\varphi_i(b)$.

Proof Since $\varepsilon_i^\star(b) = n$, whilst $\varepsilon_i(b) = m$, by the previous lemma, the assertion follows by substitution into $(*)$ and $(**)$. □

2.5.23 Take $b \in B(\infty)$ and let us write $b = b'' \otimes b_i(-n) \in B^i \otimes B_i$ and $b = b' \otimes b_i(-m) \in B^{i\star} \otimes B_i$, as before.

Lemma

(a) *The following are equivalent*
  (i) $\varphi_i(b'') \neq n$,
  (ii) $e_i^\star f_i b = f_i e_i^\star b$,
  (iii) $f_i^\star e_i b = e_i f_i^\star b$.
(b) *The following are equivalent*
  (i) $\varphi_i^\star(b') \neq m$,
  (ii) $e_i^\star f_i^{\star\star} b = f_i^{\star\star} e_i^\star b$,
  (iii) $f_i^\star e_i^{\star\star} b = e_i^{\star\star} f_i^\star b$.

Proof For (i) $\Leftrightarrow$ (ii) of (a), write $b = b'' \otimes b_i(-n) \in B^i \otimes B_i$. By definition $e_i^\star$ always enters the right hand factor and so (ii) holds exactly when f_i enters the same factor for both terms. Now f_i enters the left hand factor of b (resp. of $e_i^\star b$) if and only if $\varphi_i(b'') > n$ (resp. $\varphi_i(b'') > n - 1$). Thus commutation fails exactly when $\varphi_i(b'') = n$. The proof of (i) $\Leftrightarrow$ (iii) in (a) is similar. Finally (b) obtains from (a) by translating all arguments by $\star$. □

2.5.24 Unfortunately the above result is not free of the presentation of b. The following is a weaker result independent of presentation.

Corollary *If $b \in B(\infty)$ such that $f_i b \neq f_i^\star b$ (or equivalently $f_i^\star b \neq f_i^{\star\star} b$), then for all $k \in \mathbb{N} - \{0\}$ we have*

(i) $e_i^\star f_i b = f_i e_i^\star b$,
(ii) $f_i^\star e_i b = e_i f_i^\star b$,
(iii) $e_i^\star f_i^{\star\star} b = f_i^{\star\star} e_i^\star b$,
(iv) $f_i^\star e_i^{\star\star} b = e_i^{\star\star} f_i^\star b$.

Proof For (i) and (ii) observe that $\varphi_i(b'') \leq n \Leftrightarrow \varphi_i(b) + \varepsilon_i^\star(b) = 0$ and apply 2.5.22 to the lemma. For (iii) and (iv) observe that $\varphi_i^\star(b') \leq m \Leftrightarrow \varphi_i^\star(b) + \varepsilon_i(b) = \varphi_i(b) + \varepsilon_i^\star(b) = 0$, and apply 2.5.22 to the lemma. □

2.5.25

Theorem $\star$ *is an involution. In particular,* $f_i b = f_i^{\star\star} b$ *for all* $b \in B(\infty)$.

Proof We prove this by contradiction. Given $\lambda \in \mathbb{N}\pi$, let $|\lambda|$ denote the sum of its coefficients. Suppose that there exists $b \in B(\infty)$ such that $f_i b \neq f_i^{\star\star} b$ and choose this b to be minimal with respect to $|-\mathrm{wt}(b)|$.

If $f_i b = f_i^\star b$ then $f_i b = f_i^\star b = f_i^{\star\star} b$ by 2.5.22, so under our hypothesis $f_i b \neq f_i^\star b$. Then by 2.5.24 we have $e_i^\star f_i b = f_i e_i^\star b$ and $e_i^\star f_i^{\star\star} b = f_i^{\star\star} e_i^\star b$. Also by 2.5.15 we have that $e_j^\star f_i b = f_i e_j^\star b$ and $e_j^\star f_i^{\star\star} b = f_i^{\star\star} e_j^\star b$ for all $j \in I - \{i\}$.

Now if $j \in I$ is such that $e_j^\star b \neq 0$, then $|-\mathrm{wt}(e_j^\star b)| = |-\mathrm{wt}(b) - \alpha_j| = |-\mathrm{wt}(b)| - 1$. So by our assumption on the minimality of $|-\mathrm{wt}(b)|$ we have $f_i e_j^\star b = f_i^{\star\star} e_j^\star b$. But then by the previous paragraph we have that $e_j^\star f_i b = f_i e_j^\star b = f_i^{\star\star} e_j^\star b = e_j^\star f_i^{\star\star} b$. By applying $f_j^\star$ to both sides of this equation we obtain $f_i b = f_i^{\star\star} b$, which contradicts our assumption.

Hence we are reduced to the case that $b \in B(\infty)^{\mathscr{E}^\star} = \{b_\infty\}$, by 2.5.16. Yet $f_i b_\infty = f_i^\star b_\infty = f_i^{\star\star} b_\infty$ trivially. This contradiction proves that $f_i b = f_i^{\star\star} b$, for all $b \in B(\infty)$. □

2.5.26 The previous analysis shows that the properties of $B(\infty)$ established above, namely that $B_J(\infty)$ is independent of the choice of J, that $B(\infty) \otimes S_\lambda$ admits a strict normal subcrystal $B(\lambda)$ of highest weight λ, and that $B(\infty)$ admits a non-trivial involution $\star$ such that the induced crystal structure almost commutes (2.5.15, 2.5.24) with the original crystal structure, all follow from the upper normality of $B_J(\infty)$ and the decompositions $B(\infty) = B^i \otimes B_i : i \in I$. However even upper normality is apparently rather non-trivial as the following example indicates.

Example Take $T = \{\alpha_1, \alpha_2\}$ of type A_2, that is $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Take J with $i_j = 1$, if j is odd and $i_j = 2$, if j is even. Then $\cdots b_{i_4}(-m_4) \otimes b_{i_3}(-m_3) \otimes b_{i_2}(-m_2) \otimes b_{i_1}(-m_1) \in B_J(\infty)$ if and only if $m_3 \leq m_2$ and $m_j = 0$, for $j > 3$. One might think therefore that $B_J(\infty)$ is the subcrystal of $B_1 \otimes B_2 \otimes B_1$ generated by $b_\infty := b_1(0) \otimes b_2(0) \otimes b_1(0)$. However this is false! Consider $b = b_1(-1) \otimes b_2(-1) \otimes b_1(0)$, which belongs to $\mathscr{F} b_\infty$. Obviously $e_2 b = b_1(-1) \otimes b_2(0) \otimes b_1(0)$. This does not belong to $\mathscr{F} b_\infty$. Moreover $e_2^2 b = 0$ and also $e_1 e_2 b = 0$. Thus $b \in (\mathscr{E}\mathscr{F} b_\infty)^{\mathscr{E}}$. In particular the crystal generated by b_∞ in $B_1 \otimes B_2 \otimes B_1$ is not a highest weight crystal. By contrast if we set $b' = b_2(0) \otimes b_1(-1) \otimes b_2(-1) \otimes b_1(0)$, then e_2 goes into the first place of b' and so $e_2 b' = 0$. The point is that the crystal embedding theorem actually shows that $B(\infty)$ is a subcrystal of $b_\infty \otimes B_1 \otimes B_2 \otimes B_1$, which is slightly different from $B_1 \otimes B_2 \otimes B_1$. Instead of carrying b_∞ in the left hand factor it is enough to put $b_2(0)$ on the left. In general if $\dim \mathfrak{g}_A < \infty$, we may just take $|\Delta^+|$ terms in J determined as we shall see by any reduced decomposition of the longest element w_0 of the Weyl group W) as long as we carry b_∞ in the extreme left hand side. Alternatively one can replace b_∞ by $\bigotimes_{i \in I} b_i(0)$, with the product taken in any order. (This was not possible in the original Kashiwara theory since he took the elementary crystals to have infinite extent in both directions.)

3 Further Properties of $B(\infty)$

3.1 Character Formulae

3.1.1 Surprisingly it is extremely difficult to calculate $\operatorname{ch} B(\infty)$ especially when one considers that $B(\infty)$ is supposed to represent the dual of a Verma module of highest weight zero, and that the character formula of the latter is easily shown to be a product over the negative roots of the Lie algebra. One consequence of this product formula is nevertheless easy to prove directly for $\operatorname{ch} B(\infty)$ and we start with this. Recall the translated action of W on P defined by extending the formula

$$s_i.\lambda := s_i\lambda - \alpha_i, \quad \forall i \in I,$$

to all $w \in W$.

Lemma *For all* $w \in W$, *one has*

$$w.\operatorname{ch} B(\infty) = (-1)^{l(w)} \operatorname{ch} B(\infty).$$

Proof Recall $B^i : i \in I$ defined in 2.5.7. It is clear from the construction in 2.5.7 that

$$B^i = \varinjlim_{\lambda \in P^+ | \alpha_i^\vee(\lambda)=0} \big(T_{-\lambda}\big(B(\lambda)\big)\big)$$

where $T_{-\lambda}$ signifies translating weights by $-\lambda$. Since $B(\lambda)$ is a normal crystal, it is W invariant (see 2.2.4). Yet $\alpha_i^\vee(\lambda) = 0$, so $\operatorname{ch} T_{-\lambda} B(\lambda)$ is s_i invariant and hence so is $\operatorname{ch}\ B^i$.

By Lemma 2.5.11 one can write $B(\infty)$ as a Cartesian product

$$B(\infty) = B^i \times B_i,$$

where moreover $\operatorname{wt}(b_1, b_2) = \operatorname{wt} b_1 + \operatorname{wt} b_2$, $\forall b_1 \in B^i$, $b_2 \in B_i$. Hence $\operatorname{ch} B(\infty) = \operatorname{ch} B^i \operatorname{ch} B_i$. On the other hand it is clear that

$$\operatorname{ch} B_i = \big(1 - e^{-\alpha_i}\big)^{-1},$$

which satisfies

$$s_i.\operatorname{ch} B_i = -\operatorname{ch} B_i.$$

Combining the above observations the conclusion of the lemma results. □

3.1.2 Through a clever use of the path model, Littelmann obtained a “combinatorial character formula” for $B(\lambda) : \lambda \in P^+$ defining the family $\mathscr{F}$. We simply quote his result, an exposition of the proof of which we gave in [16, 16.11, 16.12]

Theorem (Littelmann)

$$\mathrm{ch}\, B(\lambda) = \frac{\sum_{w\in W}(-1)^{l(w)}e^{w.\lambda}}{\sum_{w\in W}(-1)^{l(w)}e^{w.0}}.$$

3.1.3 From 3.1.2 and the presentation of $B(\infty)$ as a direct limit (2.5.6) one obtains the

Corollary

$$\mathrm{ch}\, B(\infty) = \frac{1}{\sum_{w\in w}(-1)^{l(w)}e^{w.0}}.$$

3.1.4 Through the representation theory of finite dimensional simple highest weight modules for semisimple Lie algebras, Weyl first obtained an expression for the denominator occurring in 3.1.2 as a product over the negative roots. This was generalized by Kac [20, Chap. 7] using ideas of Bernstein–Gelfand–Gelfand to $\mathfrak{g}_A$, when A is symmetrizable, a key point being the use of the Casimir invariant. It is much more difficult to prove the Weyl denominator formula in general, though this was achieved independently by Kumar [27] and Mathieu [32]. Combined with 3.1.2 we obtain the formula for $\mathrm{ch}\, B(\infty)$ described in 2.4.3.

3.1.5 Littelmann's combinatorial character formula carries over to the Borcherds case [17, Sect. 9]. However, except when A is symmetrizable, it is not known if the analogue of the Weyl denominator formula holds.

3.2 Highest Weight Elements

3.2.1 Let $\mathbb{F}_A = \{B(\lambda) : \lambda \in P^+\}$ denote the closed family described in 2.5. Fix $\lambda, \mu \in P^+$ and set $B^\lambda(\mu) = \{b \in B(\mu) | e_i^{\alpha_i^\vee(\lambda)+1} b = 0,\ \forall i \in I\}$.

Lemma *One has*

$$\big(B(\lambda) \otimes B(\mu)\big)^{\mathscr{E}} = \big\{b_\lambda \otimes b | b \in B^\lambda(\mu)\big\}.$$

Proof Suppose $e_i(b_1 \otimes b_2) = 0$. If e_i goes in the right hand factor, then $e_i b_2 = 0$ so $\varepsilon_i(b_2) = 0$, by upper normality of $B(\mu)$. Yet $\varphi_i(b_1) \geq 0$, by the lower normality of $B(\lambda)$, which by the hypothesis contradicts the tensor product rule. Thus we can assume that e_i always enters the left hand factor. This forces $b_1 = b_\lambda$ and $\varepsilon_i(b_2) \leq \varphi_i(b_\lambda) = \alpha_i^\vee(\lambda),\ \forall i \in I$, that is $b_2 \in B^\lambda(\mu)$. Upper normality of $B(\mu)$ concludes the proof. □

Remark Since the proof only requires of $B(\mu)$ to be upper normal, it holds with $B(\mu)$ replaced by $B(\infty)$.

3.2.2 We remark that in Littelmann's path model, the paths lying in $B^{\lambda}(\mu)$ are exactly those lying entirely in the closure of the dominant chamber.

Take $b \in B^{\lambda}(\mu)$ and $\nu := \lambda + \mathrm{wt}\, b \in P^{+}$. It is not obvious that the crystal generated by b is a highest weight crystal or that the resulting crystal is isomorphic to $B(\nu)$. Already the case $b = b_{\mu}$ is quite difficult. The general case established by Littelmann [28] is significantly more complicated. It leads to the following decomposition theorem for the tensor product of elements of $\mathbb{F}_A$ valid in the not necessarily symmetrizable case. Kashiwara had obtained this result in the symmetrizable case by taking a $q \to 0$ limit.

Theorem *Fix $\lambda, \mu \in P^{+}$. One has the crystal decomposition*

$$B(\lambda) \otimes B(\mu) = \coprod_{b \in B^{\lambda}(\mu)} B(\lambda + \mathrm{wt}\, b).$$

Remark One might compare this to the corresponding Jordan-Holder series of tensor products in the family $\{V(\lambda) | \lambda \in P^{+}\}$ of simple highest weight $U(\mathfrak{g}_A)$ modules, known to be a direct sum in the symmetrizable case. Choose $\lambda, \mu, \nu \in P^{+}$ and set

$$V^{\mu}(\nu)_{\lambda-\mu} := \left\{ a \in V(\nu)_{\lambda-\mu} | x_{\alpha_i}^{\alpha_i^{\vee}(\mu)+1} a = 0, \ \forall i \in I \right\}.$$

Observe that $v_{\lambda} \otimes v_{-\mu}$ is a cyclic vector for $V(\lambda) \otimes V(\mu)^{*}$ under the diagonal action. Thus the map $\varphi \mapsto \varphi(v_{\lambda} \otimes v_{-\mu})$ of $\mathrm{Hom}_{\mathfrak{g}}(V(\lambda) \otimes V(\mu)^{*}, V(\nu))$ into $V(\nu)$ is injective. It may be shown to have image $V^{\mu}(\nu)_{\mu-\nu}$ irrespective of whether A is symmetrizable. Various authors have claims on this result. Zelobenko and Kostant for $\mathfrak{g}_A$ semisimple and Mathieu for the general case. A proof can be found in [11, 6.3.10] though as it is presented it is not obvious that it holds in the generality claimed here.

3.2.3 The result in 3.2.1 gives [13, Sect. 5] an interpretation of the Kashiwara involution which at first seemed rather surprising. Recall (2.5.9) that we have a strict embedding $B(\lambda) \hookrightarrow B(\infty) \otimes S(\lambda), \ \forall \lambda \in P^{+}$. Recall 3.2.1.

Proposition *Take $b \in B(\infty)$. Then for all $\lambda \in P^{+}$, one has $b \in B^{\lambda}(\infty)$ if and only if $b^{\star} \otimes s_{\lambda} \in B(\lambda)$.*

Proof Take $b \in B(\infty)$. By 2.5.7 there exist $n \in \mathbb{N}, i_1, \ldots, i_n \in J, m_1, \ldots, m_n \in \mathbb{N}$ such that

$$b = b_{\infty} \otimes b_{i_n}(-m_n) \otimes b_{i_{n-1}}(-m_{n-1}) \otimes \cdots \otimes b_{i_1}(-m_1) \in B(\infty) \otimes B_{i_n} \otimes \cdots \otimes B_{i_1}.$$

Through the Kashiwara function one checks that

$$\varepsilon_i(b) = \max_{s | i_s = i} \left\{ 0, m_s + \sum_{t > s} \alpha_i^{\vee}(\alpha_{i_t}) m_t \right\}.$$

Thus by 3.2.1 we conclude that

$$b \in B^{\lambda}(\infty) \quad \Leftrightarrow \quad \alpha_{i_s}^{\vee}(\lambda) \geq m_s + \sum_{t>s} \alpha_{i_s}^{\vee}(\alpha_{i_t})m_t, \quad \forall s = 1, \dots, n.$$

Through the crystal structure on $B(\infty)$ defined by the starred operators we may write every element of $B(\infty)$ uniquely in the form

$$b = f_{i_1}^{\star m_1} f_{i_2}^{\star m_2} \cdots f_{i_n}^{\star m_n} b_{\infty},$$

with

$$e_{i_j}^{\star} f_{i_{j+1}}^{\star m_{j+1}} \cdots f_{i_n}^{\star m_n} b_{\infty} = 0. \tag{$*$}$$

Applying 2.5.14 we obtain

$$b = b_{\infty} \otimes b_{i_n}(-m_n) \otimes \cdots \otimes b_{i_1}(-m_1),$$

as above. (Of course the m_j cannot be arbitrary as $(*)$ must be satisfied.)

Set $F_s = f_{i_s}^{m_s} \cdots f_{i_n}^{m_n} : s = 1, 2, \dots, n$, with $F_{n+1} = 1$. By definition of $\star$ we have $b^{\star} = F_1 b_{\infty}$. Now by 2.5.9 one has $b^{\star} \otimes s_{\lambda} \in B(\lambda)$, if and only if $F_1(b_{\infty} \otimes s_{\lambda}) = F_1 b_{\infty} \otimes s_{\lambda}$, that is the f_j all enter the first factor. Since $\varepsilon_i(s_{\lambda}) = -\alpha_i^{\vee}(\lambda), \ \forall i \in I$, the above holds if and only if

$$\varphi_{i_s}(F_{s+1}b_{\infty}) \geq -\alpha_{i_s}^{\vee}(\lambda) + m_s, \quad \forall s = 1, 2, \dots, n.$$

On the other hand $(*)$ translates to give $e_{i_s} F_{s+1} b_{\infty} = 0$. Since $B(\infty)$ is upper normal, this is equivalent to $\varepsilon_{i_s}(F_{s+1}b_{\infty}) = 0$.

Thus $\varphi_{i_s}(F_{s+1}b_{\infty}) = \alpha_{i_s}^{\vee}(F_{s+1}b_{\infty}) = -\sum_{t>s} \alpha_{i_s}^{\vee}(\alpha_{i_t})m_t$, for all $s = 1, 2, \dots, n$. These are exactly the conditions that $b \in B^{\lambda}(\infty)$. This proves the proposition. □

3.2.4 Since $S(\lambda)$ is a one element crystal we may identify $B(\lambda)$ with a subset of $B(\infty)$ (with the understanding that weights are translated). By this slight abuse of notation we obtain the

Corollary $B^{\lambda}(\mu) = B(\mu) \cap B(\lambda)^{\star}$.

3.2.5 The above may be compared to the following result for $U(\mathfrak{g}_A)$ modules. Take a triangular decomposition $\mathfrak{g}_A = \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is spanned by the positive (resp. negative) root vectors. Set $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$ which is a Borel subalgebra.

The dual $\delta M(0)$ of a Verma module of highest weight 0 is defined to be $\mathfrak{h}$ locally finite elements of $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{b})^*$, given a left module structure through the Chevalley antiautomorphism κ. The latter may be identified with $S(\mathfrak{n}^-)$ as a $U(\mathfrak{g})$-algebra, that is as an algebra in which $\mathfrak{g}$ acts by derivations.

Now $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{b}$ identifies with $U(\mathfrak{n}^-)$ and so the left action of $U(\mathfrak{g})$ on $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{b}$ restricts to a left action of $\mathfrak{n}^-$ which is just left multiplication by $\mathfrak{n}^-$

and defines a left $\mathfrak{n} = \kappa(\mathfrak{n}^-)$ structure on $\delta M(0)$, which is the restriction of the left $\mathfrak{g}$ action. Right multiplication of $U(\mathfrak{n}^-)$ by $\mathfrak{n}^-$ gives a right action of $\mathfrak{n}$ on $\delta M(0)$ commuting with the left action of $\mathfrak{n}^-$ (but not with the left action of $\mathfrak{g}$).

Given $\mu \in P^+$, let $\mathbb{C}_{-\mu}$ denote the one-dimensional $\mathfrak{b}$ module of weight $-\mu$. Analogously to 2.5.9, one may show that $\delta M(0)|_{U(\mathfrak{b})}$ admits a unique $U(\mathfrak{b})$ submodule isomorphic to $V(\mu)|_{U(\mathfrak{b})} \otimes \mathbb{C}_{-\mu}$ and moreover the latter is given through the right action of $U(\mathfrak{n})$ via the formula [8, Theorem 2.6],

$$\left\{a \in \delta M(0) | a x_{\alpha_i}^{\alpha_i^\vee(\mu)+1} = 0, \ \forall i \in I\right\}.$$

Now the Kashiwara involution $\star$ was obtained (see for example [11, Sect. 6.1]) via the $q \to 0$ limit and the seemingly trivial antiautomorphism defined as the identity on the generators $x_{\alpha_i} : i \in I$ in the quantization $U_q(\mathfrak{n}^-)$ of $U(\mathfrak{n}^-)$. This interchanges left and right action in $U(\mathfrak{n}^-)$ and so interchanges $V(\mu)|_{U(\mathfrak{b})} \otimes \mathbb{C}_{-\mu}$ with $V(\mu)^\star \otimes \mathbb{C}_{-\mu} := \{a \in \delta M(0) | x_{\alpha_i}^{\alpha_i^\vee(\mu)+1} a = 0, \ \forall i \in I\}$. On the other hand by the remark in 3.2.2, $\mathrm{Hom}_{\mathfrak{g}}(V(\lambda) \otimes V(\mu)^*, V(\nu))$ identifies with $((V(\nu) \otimes \mathbb{C}_{-\nu}) \cap (V(\mu)^\star \otimes \mathbb{C}_{-\mu}))_{\lambda-\mu-\nu}$. Making the cyclic permutation $\lambda \to \nu \to \mu$, this is equivalent to

$$\dim \mathrm{Hom}_{\mathfrak{g}}\big(V(\nu), V(\lambda) \otimes V(\mu)\big) = \dim\big((V(\lambda)^\star \otimes \mathbb{C}_{-\lambda}) \cap \big(V(\mu) \otimes \mathbb{C}_{-\mu}\big)\big)_{\nu-\lambda-\mu}.$$

Whereas by 3.2.4, the number of copies of $B(\nu)$ in $B(\lambda) \otimes B(\mu)$ is exactly

$$\mathrm{card}\left\{b \in B(\infty)_{\nu-\lambda-\mu} | b^\star \otimes s_\lambda \in B(\lambda), \ b \otimes s_\mu \in B(\mu)\right\}.$$

3.2.6 Observe that B_J is an additive semigroup with respect to component-wise addition. One can ask if $B_J(\infty)$ is a subsemigroup. This question will be discussed further in 3.6.1–3.6.5 using the method of Nakashima and Zelevinsky [33]. However they do not obtain a complete answer since a positivity hypothesis still has to be verified. For the moment we mention another possible approach.

Following the conclusion resulting from 3.2.3 $(*)$, it is enough to show that $e_i^\star b = e_i^\star b' = 0$ implies $e_i^\star(b + b') = 0$. Here the sum refers to the additive structure on B_J. Since $B(\infty)$ is upper normal (by construction) it is enough to show that $\varepsilon_i^\star(b) = \varepsilon_i^\star(b') = 0$ implies $\varepsilon_i^\star(b + b') = 0$. Now suppose that there exists on B_J an analogue of the Kashiwara function $b \mapsto r_i^{\star k}(b)$, which is additive *with respect to the additive structure on* B_J such that $\varepsilon_i^\star(b) = \max_k(r_i^{\star k}(b))$. Assume further (as in the case of the Kashiwara function) that $r_i^{\star k}(b) = 0$, for all $k \in \mathbb{N}^+$ sufficiently large. Then $\varepsilon_i^\star(b) = 0$ (resp. $\varepsilon_i^\star(b') = 0$) implies $r_i^{\star k}(b) \le 0$ (resp. $r_i^{\star k}(b') \le 0$), for all $k \in \mathbb{N}^+$. Then $r_i^{\star k}(b+b') \le 0$, for all $k \in \mathbb{N}^+$, forcing through the second property that $\varepsilon_i^\star(b + b') = 0$, as required.

A further anticipated property of $B(\infty)$ which we failed to prove can be expressed through the following question.

Suppose

$$b = b_\infty \otimes b_{i_n}(-m_n) \otimes \cdots \otimes b_{i_1}(-m_1) \in B(\infty),$$

with $m_n > 0$. Then does

$$b = b_\infty \otimes b_{i_n}\big(-(m_n - 1)\big) \otimes \cdots \otimes b_{i_1}(-m_1) \in B(\infty)$$

hold?

3.3 Braid Relations

3.3.1 For each $i \in I$, set $\mathscr{E}_i = \bigcup_{n\in\mathbb{N}} e_i^n$, $\mathscr{F}_i = \bigcup_{n\in\mathbb{N}} f_i^n$, where by convention $e_i^0 = f_i^0 = 1$.

Given $w \in W$, set $r = l(w)$ and fix a reduced decomposition $\underline{w} = s_{i_r} s_{i_{r-1}} \cdots s_{i_1}$ of w. Set $\mathscr{E}_{\underline{w}} = \mathscr{E}_{i_r}\mathscr{E}_{i_{r-r}} \cdots \mathscr{E}_{i_1}$, $\mathscr{F}_{\underline{w}} = \mathscr{F}_{i_r}\mathscr{F}_{i_{r-1}} \cdots \mathscr{F}_{i_1}$.

Now let B be a crystal and take $b \in B$. One can ask if $\mathscr{E}_{\underline{w}}b$ and $\mathscr{F}_{\underline{w}}b$ depend only on w. One can hardly expect this to be true in an arbitrary crystal. However we show that this does hold for crystals in the family $\mathbb{F}_A$ defined in 2.5 and by taking the limit for $B(\infty)$.

3.3.2 Suppose we replace e_i (resp. f_i) by the corresponding root vector x_{α_i} (resp. $x_{-\alpha_i}$). Since in the enveloping algebra $U(\mathfrak{g})$ we have linear structure at our disposal, the natural analogues of $\mathscr{E}_i$, $\mathscr{F}_i$ are $\mathbb{C}[x_{\alpha_i}]$, $\mathbb{C}[x_{-\alpha_i}]$ respectively. Then we may define analogues $U_{\underline{w}}$ (resp. $U_{\underline{w}}^-$) of $\mathscr{E}_{\underline{w}}$ (resp. $\mathscr{F}_{\underline{w}}$) as products in $U(\mathfrak{n})$ (resp. $U(\mathfrak{n}^-)$) of $\mathbb{C}[x_{\alpha_i}]$, $\mathbb{C}[x_{-\alpha_i}]$ respecting the order defined by $\underline{w}$. In this there are various ways to show that these subspaces depend only on w. For example, for each $\lambda \in P^+$, the subspace $U_{\underline{w}}^- v_\lambda \subset V(\lambda)$ is just the $U(\mathfrak{n})$ module generated by $v_{w\lambda}$ [11, Sect. 4.4]. This approach does not quite work in the crystal framework. Indeed by 2.2.4, there exists a unique element $b_{w\lambda} \in B(\lambda)$ of weight $w\lambda$, trivially $b_{w\lambda} \in \mathscr{F}_{\underline{w}}b_\lambda$ and via the use of $B(\infty)$ one may show that $\mathscr{F}_{\underline{w}}b_\lambda$ is $\mathscr{E}$ stable. Yet it is false that the resulting inclusion $\mathscr{E}b_{w\lambda} \subset \mathscr{F}_{\underline{w}}b_\lambda$ is an equality. (A simple example occurs in the adjoint representation of $\mathfrak{sl}(3)$.)

Nevertheless one can show that $\mathscr{F}_{\underline{w}}b_\lambda$ depends only on w. In the Kashiwara theory, this results from the fact that $U_w^- v_\lambda$ admits a basis formed from the subset of the global basis defined by $\mathscr{F}_{\underline{w}}b_\lambda$. Naturally this is extremely complicated and also requires A to be symmetrizable.

Littelmann gives a purely combinatorial proof that $\mathscr{F}_{\underline{w}}b_\lambda$ depends only on w. Indeed in his model, the subset $\mathscr{F}_{\underline{w}}b_\lambda$ is given by piecewise linear paths in $\mathbb{Q}P$ described by all Bruhat sequences beginning at the identity and ending in w, satisfying an integrability condition depending on λ. Here we admit this result (for an exposition—see [16, Sect. 16]) and show it implies that $\mathscr{F}_{\underline{w}}b : b \in B(\lambda)$ also only depends on w by an argument which is also due to Littelmann, but which does not involve paths but only the tensor product rule.

3.3.3 Given a monomial $f \in \mathscr{F}$, an element $f' \in \mathscr{F}$ in which some of the factors in f are deleted is called a submonomial.

The proof of the assertion in 3.3.1 results from Theorem 3.2.2 and the following rather easy

Lemma *Fix* $\lambda, \mu \in P^+$, $b \in B(\lambda)$, $b' \in B(\mu)$.

(i) *For each* $f' \in \mathscr{F}_{\underline{w}}$, *there exists submonomials* $f'', f \in \mathscr{F}_{\underline{w}}$ *of* f' *such that*

$$f'(b \otimes b') = f''b \otimes fb'.$$

(ii) *For each* $f \in \mathscr{F}_{\underline{w}}$, *there exists* $f', f'' \in \mathscr{F}_{\underline{w}}$ *with* f'', f *submonomials of* f' *such that*

$$f'(b \otimes b') = f''b \otimes fb'.$$

Proof (i) is a completely obvious consequence of the tensor product rule. For (ii), set $m = \max\{0, \varphi_i(b) - \varepsilon_i(b')\}$. Then by the tensor product rule $f_i^{m+t}(b \otimes b') = f_i^m(b) \otimes f_i^t b'$, $\forall t \in \mathbb{N}$. Thus by taking an appropriate power of f_i, any power of f_i may be brought into the second factor. Then (ii) follows by induction on $l(w)$. □

3.3.4 Take $\mu \in P^+$ and $b \in B(\mu)$. By 3.2.1, there exists $\lambda \in P^+$ such that $b_\lambda \otimes b \in (B(\lambda) \otimes B(\mu))^{\mathscr{E}}$. Set $\nu = \lambda + \mathrm{wt}\, b$. By 3.2.2, this element generates the highest weight crystal $B(\nu)$ and so by 3.3.2 one has

$$\mathscr{F}_{\underline{w}_1}(b_\lambda \otimes b) = \mathscr{F}_{\underline{w}_2}(b_\lambda \otimes b), \qquad (*)$$

for any two reduced decompositions $\underline{w}_1, \underline{w}_2$ of w.

Proposition *The subset* $\mathscr{F}_{\underline{w}} b \subset B(\mu)$ *depends only on* w.

Proof Given $f \in \mathscr{F}_{\underline{w}_1}$ choose f', f'' as in the conclusion 3.3.3 (ii). By $(*)$, there exists $\tilde{f}' \in \mathscr{F}_{\underline{w}_2}$ such that $f'(b_\lambda \otimes b) = \tilde{f}'(b_\lambda \otimes b)$ and let $\tilde{f}'', \tilde{f}$ be as in the conclusion of 3.3.3 (i). Then $f''b_\lambda \otimes fb = f'(b_\lambda \otimes b) = \tilde{f}'(b_\lambda \otimes b) = \tilde{f}''b_\lambda \otimes \tilde{f}b$, forcing $fb = \tilde{f}b$, and hence $\mathscr{F}_{\underline{w}_1} b \subset \mathscr{F}_{\underline{w}_2} b$. Interchanging $\underline{w}_1, \underline{w}_2$ gives the reverse inclusion. □

3.4 The String Property

3.4.1 The $U(\mathfrak{b})$ modules $F(w\lambda) := U(\mathfrak{n})v_{w\lambda}$ discussed in 3.3.2 were studied by Demazure partly because of their geometric significance being the algebra of global sections of an invertible sheaf $\mathcal{L}_\lambda$ defined by $\lambda \in P^+$ on the Schubert variety defined by w. It is known for example that for λ regular their inclusion relations are given by Bruhat order on W. Demazure calculated [5, 6] their characters based on a lemma of Verma implying a "string property" with respect to the $\mathfrak{sl}(2)$ subalgebras defined by the simple roots. As noted later by Kac the lemma is false and so is the string property. However it was shown by Kashiwara (and later by Littelmann in his path model) that the string property does hold for the corresponding "crystals" $\mathscr{F}_{\underline{w}} b_\lambda$. This we shall prove by the Kashiwara method profiting from the fact that the main technical tool is the Kashiwara involution, which we have shown to be defined in the general, not necessarily symmetrizable, case.

3.4.2 Before going further we remark that one may try to compute the characters of the homology spaces $H_*(\mathfrak{n}, F(w\lambda))$ which are $\mathfrak{h}$ modules. This is more detailed information than the $\mathrm{ch}\, F(w\lambda)$. In [9, 10] one may find information on these homology spaces which gave the first correct proof of the Demazure character formula for large λ and in characteristic zero. Later a correct proof for all λ was given by Anderson [1] using the Steinberg module. Andersen's proof is also valid in good characteristic. The Kashiwara theory [23] gives a proof which is characteristic free, entirely elementary but far from simple. One might hope it would also describe the homology spaces $H_*(\mathfrak{n}, F(w\lambda))$ but so far no success has been reported. Kumar [25] has translated the calculation of $H_*(\mathfrak{n}, F(w\lambda))$ to a problem in sheaf cohomology.

3.4.3 Adopt the notation of 3.3.1 and set $B_w(\infty) = \mathscr{F}_{\underline{w}} b_\infty$, which as noted in 3.3.2 is independent of the reduced decomposition $\underline{w} = s_{i_r} s_{i_{r-1}} \cdots s_{i_1}$ of w. Notice we can also assume that J is chosen to be $\dots i_r, i_{r-1}, \dots, i_1$ and consequently $B_w(\infty) \subset b_\infty \otimes B_{i_r} \otimes B_{i_{r-1}} \otimes \cdots \otimes B_{i_1}$. From this one easily obtains the

Lemma *For all $w \in W$, $B_w(\infty)$ is $\mathscr{E}$ stable.*

3.4.4 Suppose $w \in W$ and $i \in I$ satisfy $s_i w < w$. Without loss of generality we can assume that $i = i_r$ in the notation of 3.4.

Lemma *If $b \in B(\infty)$, satisfies $e_i b = 0$ and $f_i^k b \in B_w(\infty)$ for some $k \in \mathbb{N}$, then $b \in B_{s_i w}(\infty)$.*

Proof By the previous lemma $b = e_i^k f_i^k b \in B_w(\infty)$. Thus we can write $b = b_\infty \otimes b_{i_r}(-m_r) \otimes \cdots \otimes b_{i_1}(-m_1)$. Since $i = i_r$, one has $0 = \varepsilon_i(b) \geq \varepsilon_i(b_i(-m_i)) = m_i$. Thus $b \in B_{s_i w}(\infty)$. □

3.4.5 We make the following preliminary to establishing the string property. The argument is due to Kashiwara [23].

Proposition *Fix $w \in W$ and $i \in I$. Suppose $b \in B(\infty)$ satisfies $e_i^\star b = 0$, $f_i^\star b \in B_w(\infty)$. Then $f_i^{\star k} b \in B_w(\infty)$, for all $k \in \mathbb{N}$.*

Proof The proof is by induction on $l(w)$. If $l(w) = 0$, the hypothesis $f_i^\star b \in B_w(\infty) = \{b_\infty\}$, cannot be satisfied, so there is nothing to prove.

Suppose $l(w) > 0$. Then there exists $j \in I$ such that $l(s_j w) < l(w)$.

Suppose $j \neq i$. Then the assertion is an immediate consequence of 2.5.15. Indeed we can write $b = f_j^t b'$, with $e_j b' = 0$. Then $e_j f_i^\star b' = 0$, whilst $f_j^t f_i^\star b' = f_i^\star b \in B_w(\infty)$. Thus $f_i^\star b' \in B_{s_j w}(\infty)$, by 3.4.4. Again $0 = e_i^\star b = f_j^t e_i^\star b'$, so $e_i^\star b' = 0$, by the injectivity of f_j. Then $f_i^{\star k} b' \in B_{s_j w}(\infty)$, for all $k \in \mathbb{N}$, by the induction hypothesis. Then $f_i^{\star k} b = f_j^t f_i^{\star k} b' \in B_w(\infty)$, for all $k \in \mathbb{N}$, as required.

Suppose $j = i$. Since $e_i^\star b = 0$, we can write $\psi_i(b) = b \otimes b_i(0)$. From now one we omit the injection ψ_i for ease of notation. Then $f_i^\star b = b \otimes b_i(-1)$.

Suppose $\varphi_i(b) \leq \varepsilon_i(b_i(-1)) = 1$. Then powers of f_i enter in the right hand factor. Thus $f_i^{k-1} f_i^\star b = b \otimes b_i(-k) = f_i^{\star k} b$. By the hypothesis $f_i^\star b \in B_w(\infty)$ and since $s_i w < w$, $B_w(\infty)$ is $\mathscr{F}_i$ stable, hence $f_i^{\star k} b \in B_w(\infty)$, for all $k \in \mathbb{N}$.

Suppose $\varphi_i(b) > \varepsilon_i(b_i(-1)) = 1$. Write $b = f_i^t b'$ with $e_i b' = 0$. Then $\varphi_i(b') = \varphi_i(b) + t > t+1$. Consequently $e_i(b' \otimes b_i(-1)) = 0$, whilst $f_i^t(b' \otimes b_i(-1)) = b \otimes b_i(-1) = f_i^\star b \in B_w(\infty)$. Consequently $b' \otimes b_i(-1) \in B_{s_i w}(\infty)$. Yet $e_i^\star b' = 0$ and $f_i^\star b' = b' \otimes b_i(-1) \in B_{s_i w}(\infty)$, so by the induction hypothesis $f_i^{\star k} b' \in B_{s_i w}(\infty)$, for all $k \in \mathbb{N}$. Then

$$f_i^{\star k} b = f_i^{\star k} f_i^t b' = f_i^{\star k} f_i^t \big(b' \otimes b_i(0)\big) = f_i^{\star k} \big(f_i^t b' \otimes b_i(0)\big) = f_i^t b' \otimes f_i^k b_i(0).$$

Apply powers of e_i to this expression. Through the last part of Theorem 2.5.7 as noted in the proof of 2.5.11, e_i^u will enter the right hand factor for some $u : 0 \leq u \leq k$ and from then on powers of e_i will enter the left hand factor. Thus we can write the expression as $f_i^{t+u}(b' \otimes f_i^{k-u} b_i(0))$. The latter equals $f_i^{t+u}(f_i^{\star(k-u)} b') \in f_i^{t+u} B_{s_i w}(\infty) \subset B_w(\infty)$. This gives the required result. □

3.4.6 Suppose $b \in B_w(\infty)$. Through the presentation given in 3.4.3, we can write $b = f_{i_r}^{m_r} \cdots f_{i_1}^{m_1} b_\infty$ with $e_{i_s} f_{i_{s-1}}^{m_{s-1}} \cdots f_{i_1}^{m_1} b_\infty = 0$, for all $s = 1, 2, \ldots, r$. Then $b^\star \in B_{w^{-1}}(\infty)$ by 2.5.14. This leads to the following

Theorem *Fix $w \in W$ and $i \in I$. If $b \in B(\infty)$ satisfies $f_i b \in B_w(\infty)$, then $f_i^k b \in B_w(\infty)$, for all $k \in \mathbb{N}$.*

Proof By 3.4.3 we can assume $e_i b = 0$. By definition of $\star$ one has $(e_i b)^\star = e_i^\star b^\star$ and $(f_i^k b)^\star = f_i^{\star k} b^\star$. Thus the hypotheses of 3.4.5 are satisfied with replacing b by $b^\star$ and w by w^{-1}. From its conclusion $f_i^{\star k} b^\star \in B_{w^{-1}}(\infty)$, for all $k \in \mathbb{N}$ and applying $\star$ we obtain $f_i^k b \in B_w(\infty)$, as required. □

3.4.7 Through the strict embedding $B(\lambda) \hookrightarrow B(\infty) \otimes S_\lambda$ obtained in 2.5.9 we obtain the

Corollary *Fix $w \in W$, $i \in I$, $\lambda \in P^+$. If $f_i b \in B_w(\lambda)$, then $f_i^k b \in B_w(\lambda)$, for all $k \in \mathbb{N}$.*

3.4.8 The above result is just the string property of $B_w(\lambda)$, which Demazure had attempted to prove for the $U(\mathfrak{b})$ module $F(w\lambda)$. Its interest lies in the fact that it leads inductively to the formula for $\operatorname{ch} B_w(\lambda)$ described below.

3.4.9 Fix $i \in I$. For each $\lambda \in P$, set

$$\Delta_i e^\lambda = \big(1 - e^{-\alpha_i}\big)^{-1} \big(e^\lambda - e^{s_i \lambda - \alpha_i}\big)$$

which one may check belongs to $\mathbb{Z}P$. Hence Δ_i is a $\mathbb{Z}$ linear endomorphism of $\mathbb{Z}P$, called the Demazure operator.

Call $S \subset B(\lambda)$ an i-string if $S = \mathscr{F}_i s$, for some $s \in B(\lambda)$ satisfying $e_i s = 0$. Through the normality of $B(\lambda)$, one checks that

$$\operatorname{ch} S = \Delta_i e^{wt(s)}.$$

On the other hand one may also check that $\Delta_i^2 = \Delta_i$. Thus

$$\operatorname{ch} S = \Delta_i \operatorname{ch} S.$$

These two marvellous formulae combined with the conclusion of 3.4.7 give the following

Lemma *Fix $w \in W$, $i \in I$ such that $s_i w > w$. Then for all $\lambda \in P^+$, one has*

$$\operatorname{ch} B_{s_i w}(\lambda) = \Delta_i \operatorname{ch} B_w(\lambda).$$

Proof It remains to observe that the condition $s_i w > w$, implies that $B_{s_i w}(\lambda)$ is $\mathscr{F}_i$ stable, thus a disjoint union of i-strings each of which either already lie in $B_w(\lambda)$ or whose intersection with $B_w(\lambda)$ consists of the unique element annihilated by e_i. □

3.4.10 This result already has an interesting corollary resulting from the independence of $\mathscr{F}_{\underline{w}} b_\lambda = B(w\lambda)$ on the reduced decomposition $\underline{w}$ of w.

Corollary *The $\Delta_i : i \in I$ satisfy the Coxeter relations.*

Remark Of course this may also be checked by explicit computation; but be warned that Demazure [6] has described the calculation in G_2 as being "épouvantable".

3.4.11 Suppose $|W| < \infty$. Then it admits a unique longest element w_0. It follows from 3.4.10 and the idempotence of the Δ_i, that $\Delta_i \Delta_{w_0} = \Delta_{w_0}$, for all $i \in I$.

Given $a \in \mathbb{Z}P$ one easily checks that $\Delta_i a = a$, if and only if $s_i a = a$. In particular we conclude that $\operatorname{ch} B(\lambda) = \Delta_{w_0} e^\lambda$ is W invariant. From this and the particular form of Δ_{w_0}, Demazure [6] noted that $\Delta_{w_0} e^\lambda$ is just the Weyl character formula. Unfortunately the corresponding result and argument fails in the case that W is infinite.

3.4.12 The algebra with generators $T_i : i \in I$ and relations $T_i^2 = T_i : i \in I$, together with the Coxeter relations is called the singular Hecke algebra. So far we have met two examples. The first is given by the $\mathscr{F}_i : i \in I$ acting on $B(\infty)$, the second by the Demazure operators $\Delta_i : i \in I$ acting on $\mathbb{Z}P$. One may remark that as a consequence any monomial $T_{i_1} T_{i_2} \cdots T_{i_n}$ can be written as T_y where $T_y = T_{j_1} T_{j_2} \cdots T_{j_m}$ with $s_{j_1} s_{j_2} \cdots s_{j_m}$ a reduced decomposition of y.

3.5 Combinatorial Demazure Flags

3.5.1 Some 25 years ago, I suggested to my then Ph.D. student P. Polo that the category of Demazure modules may admit a tensor structure, namely that for all $\lambda, \mu \in P^+$, $y, w \in W$, the tensor product $F(y\lambda) \otimes F(w\mu)$ admits a Demazure flag, that is a $U(\mathfrak{b})$ filtration whose quotients are again Demazure modules. This turned out to be false; but it appears to hold if we take $y = Id$. Indeed for $\mathfrak{g}$ semisimple (equivalently if W is finite) Polo checked most cases. Then somewhat in a spirit of competition O. Mathieu proved this to be true for all $\mathfrak{g}$ semisimple and in all characteristics—see [21] for an exposition. The proof occupies an entire book! The method which depended on a universality property of Demazure modules broke down for arbitrary Kac–Moody Lie algebras. However I later [15] showed it still hold in all characteristics for any $\mathfrak{g}_A$ with A symmetric and simply-laced. This needed the Kashiwara $q \to 0$ limit of these modules over the quantized enveloping algebra (requiring A symmetrizable) together with a positivity property in the multiplication of canonical bases due to Lusztig requiring A simply-laced. An essential though elementary step in the proof was a corresponding decomposition for "Demazure crystals" which holds for all A. This result is described below.

3.5.2 For all $w \in W$, $\lambda \in P^+$, set $B_w^\lambda(\infty) = B^\lambda(\infty) \cap B_w(\infty)$. Given $b \in B_w^\lambda(\infty)$ set

$$\mathscr{F}_{w,b}^\lambda := \left\{ f \in \mathscr{F} \,|\, f(b_\lambda \otimes b) \subset b_\lambda \otimes B_w(\infty) \right\}.$$

Lemma

$$b_\lambda \otimes B_w(\infty) = \coprod_{b \in B_w^\lambda(\infty)} \mathscr{F}_{w,b}^\lambda (b_\lambda \otimes b).$$

Proof The inclusion $\supset$ is by construction. Conversely given $b_\lambda \otimes b' \in B_w(\infty)$, then $\mathscr{E}(b_\lambda \otimes b') = b_\lambda \otimes \mathscr{E} b'$, which by 3.4.3 contains an element of the form $b_\lambda \otimes b : b \in B_w^\lambda(\infty)$ such that $b_\lambda \otimes b' \in \mathscr{F}(b_\lambda \otimes b)$. Thus $b_\lambda \otimes b' \in \mathscr{F}_{w,b}^\lambda(b_\lambda \otimes b)$ by definition of $\mathscr{F}_{w,b}^\lambda$. Finally the union is disjoint by Theorem 3.2.2. □

3.5.3 Of course 3.5.2 does not say too much as we have to be able to compute $\mathscr{F}_{w,b}^\lambda$. This is provided by the following.

Fix a reduced decomposition $\underline{w} = s_{i_1} s_{i_2} \cdots s_{i_n}$ of $w \in W$. Given $b \in B_w(\infty)$ we may write $b = b_\infty \otimes b_{i_1}(-m_1) \otimes \cdots \otimes b_{i_n}(-m_n)$ for some $n \in \mathbb{N}$, $m_j \in \mathbb{N}$, $j = 1, 2, \ldots, n$.

At this point it is worthwhile to mention that the Kashiwara function $r_i^k(b)$ is not so interesting for $i_k \neq i$, since it simply equals $-\infty$. Instead we consider the function $b \mapsto r_{i_j}^j(b)$ which always takes a finite value.

Given $b \in B_{\underline{w}}^\lambda(\infty)$, set

$$J_{\underline{w},b}^\lambda = \left\{ j \in \{1, 2, \ldots, n\} \,|\, r_{i_j}^j(b) \leq \alpha_{i_j}^\vee(\lambda) \right\}.$$

In the monoid

$$\mathscr{F}_{\underline{w}} = \mathscr{F}_{i_1}\mathscr{F}_{i_2}\cdots\mathscr{F}_{i_n}$$

suppress the $\mathscr{F}_{i_j}$, for all $j \in J^\lambda_{\underline{w},b}$. By 3.4.12 the resulting set can be written uniquely in the form

$$\mathscr{F}_{y^\lambda_{\underline{w},b}}, \quad \text{for some } y^\lambda_{\underline{w},b} \in W.$$

We remark that it is a property of Bruhat order that $y^\lambda_{\underline{w},b} \leq w$.

Theorem *For all $\lambda \in P^+$, $w \in W$, $b \in B^\lambda_w(\infty)$ one has*

(i) $\mathscr{F}^\lambda_{w,b} \supset \mathscr{F}_{y^\lambda_{\underline{w},b}}$,
(ii) $\mathscr{F}^\lambda_{w,b}(b_\lambda \otimes b) = \mathscr{F}_{y^\lambda_{\underline{w},b}}(b_\lambda \otimes b)$,
(iii) $y^\lambda_{\underline{w},b}$ *is independent of the reduced decomposition $\underline{w}$ of w.*

Remark 1 Equality may fail in (i).

Remark 2 Set $y = y^\lambda_{\underline{w},b}$. Then the right hand side of (ii) equals $B_y(\nu)$, where $\nu = \lambda + \mathrm{wt}\, b$.

Remark 3 Combined with 3.5.2 we obtain a decomposition of $b_\lambda \otimes B_w(\infty)$ as a disjoint union of the "Demazure crystals" $B_y(\nu)$. In order to obtain the corresponding decomposition for $B_w(\mu)$, we simply restrict the elements of $B^\lambda_w(\infty)$ to lie in the first factor of the image of $B(\mu)$ in $B(\infty) \otimes S_\mu$.

3.5.4 A proof of Theorem 3.5.3 is given in [16, 19.3]. A similar result in the language of paths was proved by Littelmann [30]. A key point in the above proof is that if e_i (resp. f_i) enters at the j^{th} place in b, then further powers of e_i (resp. f_i) enter at the j^{th} place or to the left (resp. right) of the j^{th} place.

3.6 Additive Structure

3.6.1 Fix a sequence J as in 2.4.2. It is clear that B_J is a semigroup under component-wise addition. One can then ask if $B_J(\infty)$ is a subsemigroup. The (over-ambitious) goal here was to find a choice of J such that $B_J(\infty)$ is free, say on generators $\{b_k\}_{k\in K}$ running over some generally infinite set K. From this we would obtain $\mathrm{ch}\, B(\infty)$ in the required form, namely

$$\mathrm{ch}\, B(\infty) = \prod_{k\in K}\left(1 - e^{wt(b_k)}\right)^{-1}.$$

By comparison with the Weyl denominator formula, we would conclude that the $\mathrm{wt}(b_k) : k \in K$ are just the set of negative roots, giving the latter a purely combinatorial interpretation. Unfortunately it seems that freeness almost never holds, though it does hold in type A for a very particular choice of J, namely for $J = \{\alpha_1, \alpha_2, \alpha_1, \alpha_3, \alpha_2, \alpha_1, \ldots\}$. Then the weights of the generators are in natural correspondence with the negative roots. However this case is very special. Notice in the above J is always an acceptable choice even if I is infinite (and countable).

3.6.2 We describe a result of Nakashima and Zelevinsky [33] which implies the required semigroup structure for $B_J(\infty)$, under a positivity hypothesis. It is disarmingly simple.

3.6.3 Fix a sequence J as above. View each $b \in B_J$ as a sequence $\underline{m} = (\ldots, m_2, m_1) : m_i \in \mathbb{N}$ and hence as a free semigroup under component-wise addition. One can therefore speak of linear forms on B_J, that is functions $\varphi : B_J \to \mathbb{Z}$ such that $\varphi(\underline{m} + \underline{m}') = \varphi(\underline{m}) + \varphi(\underline{m}'),\ \forall \underline{m}, \underline{m}' \in B_J$. Recall 3.5.3 and observe that the modified Kashiwara function $r^k_{i_k}(\underline{m})$ which we recall is given by

$$r^k_{i_k}(\underline{m}) = m_k + \sum_{j>k} \alpha^\vee_{i_k}(\alpha_{i_j}) m_j$$

is a linear form on B_J. Write $r^k_{j_k}$ simply as r_k.

Again for all k there exists a co-ordinate form on B_J, x_k defined by $x_k(\underline{m}) = m_k$. Then every linear form φ on B_J can be written as an infinite sum

$$\varphi = \sum_{k \in \mathbb{N}^+} \varphi_k x_k.$$

In this notation

$$r_k = x_k + \sum_{j>k} \alpha^\vee_{i_k}(\alpha_{i_j}) x_j.$$

For all $k \in \mathbb{N}^+$, let k_+ be the smallest $j > k$ such that $i_j = i_k$ (which exists by the definition of J). If $i_j \neq i_k$, for all $j < k$ set $k_- = 0$. Otherwise let k_- be the largest integer $< k$ such that $i_j = i_k$.

Let t^+_k, t^-_k be the linear forms on B_J defined by

$$t^+_k := r_k - r_{k_+} = x_k + \sum_{k_+ > j > k} \alpha^\vee_{i_k}(\alpha_{i_j}) x_j + x_{k_+},$$

$$t^-_k := r_{k_-} - r_k = x_{k_-} + \sum_{k > j > k_-} \alpha^\vee_{i_k}(\alpha_{i_j}) x_j + x_k.$$

The key to the Nakashima–Zelevinski theory is the family of piecewise linear operators $S_k : k \in \mathbb{N}^+$ on linear forms given by

$$S_k(\varphi) = \begin{cases} \varphi - \varphi_k\, t^+_k & : \varphi_k \geq 0 \\ \varphi - \varphi_k\, t^-_k & : \varphi_k \leq 0. \end{cases}$$

Now one has

$$(t_k^+)_j = \begin{cases} 1 & : j \in \{k_+, k\} \\ \alpha_{i_k}^\vee(\alpha_{i_j}) & : k_+ > j > k \\ 0 & : \text{otherwise.} \end{cases}$$

$$(t_k^-)_j = \begin{cases} 1 & : j \in \{k, k_-\} \\ \alpha_{i_k}^\vee(\alpha_{i_j}) & : k > j > k_- \\ 0 & : \text{otherwise.} \end{cases}$$

In particular $(t_k^+)_k = (t_k^-)_k = 1$. Thus $(\varphi - \varphi_k\ t_k^+)_k = (\varphi - \varphi_k\ t_k^-)_k = 0$. Consequently $S_k^2(\varphi) = S_k(\varphi)$, for all linear forms φ on B_J.

3.6.4 Through the operators $S_k : k \in \mathbb{N}^+$ and the co-ordinate linear forms $x_l : l \in \mathbb{N}^+$ one may generate a family $\mathcal{L} = \{S_{k_1} S_{k_2} \cdots S_{k_t} x_{k_{t+1}} : t \in \mathbb{N}, k_i \in \mathbb{N}^+\}$ of linear forms on B_J.

The positivity hypothesis on $\mathcal{L}$ is that for all $\varphi \in \mathcal{L}$ one has $\varphi(k) \geq 0$, for all k for which $k_- = 0$. The following result is due to Nakashima and Zelevinsky [33, Theorem 3.5]

Theorem *Assume $\mathcal{L}$ satisfies the positivity hypothesis. Then*

$$B_J(\infty) = \left\{\underline{m} \in B_J | \varphi(\underline{m}) \geq 0, \textit{for all } \varphi \in \mathcal{L}\right\}.$$

Proof We first show that the right hand side $B'_J(\infty)$ is stable under $\mathscr{F}, \mathscr{E}$.

Take $i \in I$, $\underline{m} \in B'_J(\infty)$ and show that $f_i\underline{m} \in B'_J$. Choose $k \in \mathbb{N}^+$, so that f_i enters at the k^{th} place, thus sending m_k to m_{k+1}. In particular $\varphi(f_i\underline{m}) = \varphi(\underline{m}) + \varphi_k \geq \varphi_k$, since $\underline{m} \in B'_J(\infty)$. Thus to show $\varphi(f_i\underline{m}) \geq 0$, it is enough to consider the case when $\varphi_k < 0$. By the positivity hypothesis this means $k_- \geq 1$.

The condition that f_i enters in the k^{th} place implies by 2.3.2 (2) that $r_k(\underline{m}) > r_{k_-}(\underline{m})$. Thus $t_k^-(\underline{m}) = r_{k_-}(\underline{m}) - r_k(\underline{m}) \leq -1$. Consequently $\varphi(f_i\underline{m}) = \varphi(\underline{m}) + \varphi_k \geq \varphi(\underline{m}) - \varphi_k t_k^-(\underline{m}) = (S_k\varphi)(\underline{m}) \geq 0$, since $\underline{m} \in B'_J(\infty)$.

Next we show that $e_i\underline{m} \in B'_J \cup \{0\}$. Suppose that x_i enters in the k^{th} place. If $m_k = 0$, then $e_i\underline{m} = 0$, so suppose that $m_k \geq 1$. Then $\varphi(e_i\underline{m}) = \varphi(\underline{m}) - \varphi_k \geq -\varphi_k$, so it is enough to consider the case when $\varphi_k > 0$. By the condition that x_i enters in the k^{th} place one has $r_k(\underline{m}) > r_{k_+}(\underline{m})$, so then $t_k^+(\underline{m}) \geq 1$. Consequently

$$\varphi(e_i\underline{m}) = \varphi(\underline{m}) - \varphi_k \geq \varphi(\underline{m}) - \varphi_k t_k^+(\underline{m}) = (S_k\varphi)(\underline{m}) \geq 0,$$

since $\underline{m} \in B'_J(\infty)$.

Finally suppose that $\underline{m} \neq \underline{0}$. Let l be the maximal index such that $m_l > 0$. Set $i = i_l$ and suppose that $e_i\underline{m} = 0$. Suppose e_i enters at the k^{th} place. Since $r_l(\underline{m}) = r_{i_l}^l(\underline{m}) = m_l > 0$ and $r_{l'}(\underline{m}) = 0$ for $l' > l$, one has $k \leq l$. Then $e_i\underline{m} = 0$, implies $m_k = 0$. Consider the co-ordinate form $\varphi = x_k$. One has $\varphi_k = 1$, so $S_k x_k = x_k - t_k^+$. Then since $\underline{m} \in B'_J(\infty)$, we must have

$$0 \leq (S_k x_k)(\underline{m}) = x_k(\underline{m}) - t_k^+(\underline{m}) = -t_k^+(\underline{m}),$$

forcing $r_k(\underline{m}) - r_{k_+}(\underline{m}) = t_k^+(\underline{m}) \leq 0$. On the other hand $t_k^+(\underline{m}) \geq 1$, by the assumption the e_i enters at the k^{th} place. This contradiction implies that $e_i\underline{m} \neq 0$.

It follows that $(B'_J(\infty))^{\mathscr{E}} = \{0\} = b_\infty$. Consequently $B'_J(\infty) = \mathscr{F} b_\infty = B_J(\infty)$, as required. □

Remark 1 This use of x_k, $S_k x_k$ does not occur in the original argument of Nakashima and Zelevinsky ([33], end of proof of Theorem 3.5). Either they forgot to include it, or there was a gap in their reasoning.

Remark 2 Showing that $k \leq l$, avoids the awkwardness of having e_i enter at an infinite place (see 2.3.5).

Remark 3 Notice that we can turn the argument of the last part around to show that $B'_J(\infty)$ (and hence $B_J(\infty)$) is upper normal! Indeed suppose $e_i\underline{m} = 0$. If e_i enters at a finite place, say at the k^{th} place, then the above reasoning gives a contradiction. Thus e_i must enter at an infinite place. This means that $\varepsilon_i(\underline{m}) = 0$. Hence we have shown that $B'_J(\infty)$ is upper normal. Indeed already in B_J (which is not upper normal) one has $\infty > \varepsilon_i(b) \geq 0, \ \forall b \in B_J$. Thus if $b \in B'_J(\infty)$ satisfies $\varepsilon_i(b) > 0$ one has $e_i b \in B'_J$ and then $\varepsilon_i(e_i(b)) = \varepsilon_i(b) - 1$ by (C2). Induction then establishes that $\varepsilon_i(b) = \max_k\{e_i^k b \neq 0\}$.

3.6.5 The result noted in Remark 3 and the discussion in 2.5.26 means that the above theory can be used to construct $\mathbb{F}_A$ without recourse to the Littelmann theory. However we still need the positivity hypothesis on $\mathcal{L}$—see 3.6.4, Nakashima and Zelevinsky [33] note that it holds in all the examples they study. However it maybe be that for a given A, there may be no choices of J for which the positivity hypothesis can be verified.

3.6.6 Assume the positivity hypothesis holds for a given A and a given J. Then we obtain the immediate corollary of Theorem 3.6.4.

Corollary $B_J(\infty)$ *is a semigroup under component-wise addition.*

Remark 4 It would be interesting to give an algorithm for finding generators.

3.6.7 A further question that one may ask is whether $S_k : k \in \mathbb{N}^+$ of 3.6.3 generate a singular Hecke algebra corresponding to an affinization of W.

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Structure and Representation Theory of Kac–Moody Superalgebras

Vera Serganova

Abstract The aim of these lecture notes is to give an introduction to the structure and representation theory of Lie superalgebras.

We start by reviewing some basic facts about finite-dimensional and Kac–Moody Lie superalgebras. Then we review the classification of Kac–Moody superalgebras of finite growth and give a survey of results about highest weight integrable representations of Kac–Moody Lie superalgebras.

In the last section we discuss the representation theory of finite-dimensional superalgebras. We formulate an analogue of a theorem of Harish-Chandra and review some geometric methods: associated variety and the Borel–Weil–Bott theorem. We omit all long and technical proofs referring to the original papers but try to explain the main ideas.

Keywords Lie superalgebra · Cartan matrix · Affine superalgebra · Degree of atypicality · Associated variety

Mathematics Subject Classification (2010) 17B67

1 Lie Superalgebras: Definitions and Examples

1.1 Introduction

The idea of supersymmetry plays an important role in modern physics. The methods of supersymmetry allow us to reduce some physics questions to questions in the theory of representations of supergroups and superalgebras. The aim of these lectures is to give an accessible introduction to the mathematical aspects of supersymmetry.

We start by reviewing some basic facts about finite-dimensional and Kac–Moody Lie superalgebras. Then we review results about highest weight integrable representations of Kac–Moody Lie superalgebras.

In the last section we discuss the representation theory of finite-dimensional superalgebras. We formulate an analogue of a theorem of Harish-Chandra and review

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some geometric methods: associated variety and the Borel–Weil–Bott theorem. We omit all long and technical proofs referring to the original papers but try to explain the main ideas.

1.2 Superalgebras and the Sign Rule

Let k be a fixed field of characteristic not equal to 2. An *superalgebra* A is a $\mathbb{Z}_2$-graded associative k-algebra, i.e., $A = A_0 \oplus A_1$ and $A_i A_j \subset A_{i+j}$. If $x \in A_0$ or $x \in A_1$, then x is called *homogeneous*. If $x \in A_0$ (resp. $x \in A_1$), we write $p(x) = 0$ (resp. $p(x) = 1$). The elements of A_0 (resp., A_1) are called *even* (resp. *odd*). If x and y are both homogeneous, then the product xy is also homogeneous, and $p(xy) = p(x) + p(y) \bmod 2$.

A *module* over a superalgebra A is a $\mathbb{Z}_2$-graded A-module. In particular, a *vector superspace* V is a k-module, i.e., a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. The dimension of a vector superspace is a pair $(m|n)$, where $m = \dim V_0$, $n = \dim V_1$.

There is a natural "change of parity" functor on the category of A-modules. We denote this functor by π. By definition for any A-module M, $\pi(M)$ is the module with shifted parity, i.e., $\pi(M)_0 = M_1$ and $\pi(M)_1 = M_0$.

All subalgebras, ideals, submodules by definition are $\mathbb{Z}_2$-graded subspaces with grading inherited from the one on the original object.

Let A be a superalgebra, and M be an A-module. Then $\mathrm{End}_{\mathsf{k}}(M)$ has a natural associative superalgebra structure with the following $\mathbb{Z}_2$-grading:

$$\mathrm{End}_{\mathsf{k}}(M)_0 = \mathrm{End}_{\mathsf{k}}(M_0) + \mathrm{End}_{\mathsf{k}}(M_1), \tag{1}$$

$$\mathrm{End}_{\mathsf{k}}(M)_1 = \mathrm{Hom}_{\mathsf{k}}(M_0, M_1) + \mathrm{Hom}_{\mathsf{k}}(M_1, M_0). \tag{2}$$

If A is associative, then the A-module structure on M defines a homomorphism $A \to \mathrm{End}_{\mathsf{k}}(M)$.

Many classical algebraic notions can be generalized to the supersetting using the following mnemonic rule:

all identities are written for homogeneous elements only, and then extended to all elements by linearity; whenever in a formula the order of two entries a and b is switched, the sign $(-1)^{p(a)p(b)}$ *appears.*

Here are some examples.

A linear operator $d \in \mathrm{End}_{\mathsf{k}}(A)$ is a *superderivation* if it satisfies the "super"-Leibniz identity

$$d(ab) = (da)b + (-1)^{p(a)p(d)} a(db).$$

A superalgebra A is *commutative* if

$$ab = (-1)^{p(a)p(b)} ba$$

for all homogeneous $a, b \in A$.

Let V be a vector superspace. Then the *symmetric superalgebra* $S(V)$ is the quotient of the tensor algebra $T(V)$ by the ideal generated by $v \otimes w - (-1)^{p(v)p(w)} w \otimes v$ for all homogeneous $v, w \in V$. It is easy to see that $S(V)$ is a commutative superalgebra. From the classical point of view we have an isomorphism $S(V) \simeq S(V_0) \otimes \Lambda(V_1)$. In particular, if $V_0 = 0$, the symmetric superalgebra $S(V)$ is the exterior algebra $\Lambda(V)$ in the usual sense.

The supertransposition formula. Let $V = V_0 \oplus V_1$ be a vector superspace, $\{e_1, \dots, e_n\}$ be its basis (the vectors $e_1, \dots, e_k$ are assumed even, and the vectors $e_{k+1}, \dots, e_n$ are assumed odd), and $\{\varphi_1, \dots, \varphi_n\}$ be the dual basis of the dual vector space V^*. If $X \in \mathrm{End}(V)$, $v \in V$, and $\varphi \in V^*$ are homogeneous, then the adjoint operator $X^* \in \mathrm{End}(V^*)$ is defined by

$$\langle X^* \varphi, v \rangle = (-1)^{p(X)p(\varphi)} \langle \varphi, Xv \rangle.$$

If the matrix of X in the basis $\{e_1, \dots, e_n\}$ has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is an arbitrary $k \times k$-matrix, D is an arbitrary $(n-k) \times (n-k)$-matrix, and B and C are rectangular matrices of the proper size, then the adjoint operator X^* has the matrix

$$\begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix}$$

in the dual basis.

1.3 Lie Superalgebras

The following definition is central for our lectures and illustrates the sign rule principle.

A *Lie superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with an even (i.e., grading preserving) linear map $[\,,\,]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, satisfying the following conditions:

(i) $[a, b] = -(-1)^{p(a)p(b)} ba$,
(ii) $[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)} [b, [a, c]]$.

Remark 1.1 One can see that in the above definition $\mathfrak{g}_0$ is a Lie algebra and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module. The linear map $[\,,\,]\colon \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$ is a homomorphism of $\mathfrak{g}_0$-modules.

If $x \in \mathfrak{g}_1$, then $[x, x]$ may not equal zero. Using the Jacobi identity for odd x,

$$\big[x, [x, x]\big] = \big[[x, x], x\big] - \big[x, [x, x]\big] = -2\big[x, [x, x]\big],$$

one gets

$$\big[x, [x, x]\big] = 0 \quad \text{for all } x \in \mathfrak{g}_1. \tag{3}$$

Example Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ is a Lie algebra, $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module, and $[\cdot,\cdot] : S^2(\mathfrak{g}_1) \to \mathfrak{g}_0$ is a homomorphism of $\mathfrak{g}_0$-modules satisfying (3). Then $\mathfrak{g}$ is a Lie superalgebra.

Any associative superalgebra A has a natural Lie superalgebra structure defined by $[a,b] = ab - (-1)^{p(a)p(b)}ba$ for all homogeneous $a, b \in A$.

Let $V = V_0 \oplus V_1$ be a vector superspace with $\dim V_0 = m$ and $\dim V_1 = n$. Then

$$\mathfrak{gl}(m|n) = \mathrm{End}_{\mathsf{k}}(V), \qquad [X,Y] = XY - (-1)^{p(X)p(Y)}YX$$

for homogeneous $X, Y \in \mathfrak{gl}(m|n)$.

Let A be a superalgebra. Then the algebra $\mathrm{Der}\,A$ of all superderivations of A is a Lie superalgebra.

In particular, let $A = S(V)$ with $\dim V = (m|n)$. Fix a basis $x_1, \dots, x_n, \xi_1, \dots, \xi_m$ of V with $p(x_i) = 0$, $p(\xi_i) = 1$. Then

$$W(m|n) = \mathrm{Der}\,S(V) = \left\{ d = \sum_{i+1}^{m} f_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} g_i \frac{\partial}{\partial \xi_i} \;\middle|\; f_i, g_i \in S(V) \right\}.$$

The Lie superalgebra $W(m|n)$ is called the *Lie superalgebra of polynomial vector fields*.

Let $\mathfrak{g}_0 = \mathfrak{sl}_2$ and $\mathfrak{g}_1 = V_1$, where V_1 is the natural two-dimensional representation of $\mathfrak{sl}_2$. Recall that $S^2(\mathfrak{g}_1)$ is isomorphic to the adjoint $\mathfrak{sl}(2)$-module. The isomorphism $S^2(\mathfrak{g}_1) \to \mathfrak{sl}(2)$ defines a Lie bracket $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$. It is an easy exercise to check that $\mathfrak{g}$ is a Lie superalgebra. We denote it by $\mathfrak{osp}(1|2)$. It has a basis

$$x, y \in \mathfrak{g}_1, \qquad [x,x], [y,y], h \in \mathfrak{g}_0$$

satisfying the following relations:

$$[h,x] = 2x, \qquad [h,y] = -2y, \qquad [x,y] = h.$$

The Lie superalgebra $\mathfrak{osp}(1|2)$ is the smallest simple Lie superalgebra which is not a Lie algebra.

Let $\mathfrak{g}$ be a Lie superalgebra, and $x \in \mathfrak{g}$. Let $\mathrm{ad}_x \in \mathrm{End}_{\mathsf{k}}(\mathfrak{g})$ be defined by

$$\mathrm{ad}_x(y) := [x,y] \quad \text{for any } y \in \mathfrak{g}.$$

It follows immediately from the super Jacobi identity that ad_x is a superderivation of $\mathfrak{g}$.

1.4 Simple Lie Superalgebras

A Lie superalgebra is *simple* if it has no nontrivial proper ideals.

Kac [19] has classified the simple finite-dimensional Lie superalgebras over an algebraically closed field k of characteristic zero. He has divided them into three types.

I	Contragredient	Basic classical: $\mathfrak{sl}(m\|n)$, $\mathfrak{osp}(m\|2n)$ Exceptional[1]: $D(2,1;a)$, $G(1\|2)$, $F(1\|3)$
II	Classical, but not contragredient	$\mathfrak{q}(n)$, $\mathfrak{p}(n)$
III	Cartan type	$W(0\|n)$, $S(n)$, $S'(n)$, $H(n)$

1.4.1 Basic Classical Lie Superalgebras

(1) Let X be an $(m+n)\times(m+n)$-matrix of the form

$$X=\begin{pmatrix} A & B\\ C & D\end{pmatrix}.$$

The *supertrace* of X is

$$\operatorname{str} X=\operatorname{tr} A-\operatorname{tr} D.$$

Define the *special linear Lie superalgebra* as

$$\mathfrak{sl}(m|n)=\left\{X\in\mathfrak{gl}(m|n)|\operatorname{str} X=0\right\}.$$

Exercise 1.2 *Show that $\mathfrak{sl}(m|n)$ is simple if and only if $m\neq n$. Otherwise its center is nontrivial, and the quotient superalgebra $\mathfrak{psl}(n|n)$ is simple for $n>1$.*

(2) Fix an even symmetric bilinear form $(\cdot,\cdot)$ on a vector superspace V, $\dim V=(m|2n)$ (in the usual sense this form is symmetric on V_0 and skew-symmetric on V_1). The *orthosymplectic* Lie superalgebra

$$\mathfrak{osp}(m|2n)=\Big\{x\in\mathfrak{gl}(m|2n)\mid (xv,w)+(-1)^{p(x)p(v)}(v,xw)=0 \\ \text{for all homogeneous } v,w\in V\Big\}$$

is simple if $m,n>0$; the even part of $\mathfrak{osp}(m|2n)$ is isomorphic to $\mathfrak{o}(m)\oplus\mathfrak{sp}(2n)$.

1.4.2 Exceptional Superalgebras

(1) The even part of $D(2,1;a)$ is isomorphic to $\mathfrak{sl}_2\oplus\mathfrak{sl}_2\oplus\mathfrak{sl}_2$, and the odd part is the outer tensor product $V_1\boxtimes V_1\boxtimes V_1$ of three copies of the standard two-dimensional $\mathfrak{sl}_2$-module V_1. Recall that $S^2(V_1)$ is isomorphic to the adjoint representation of $\mathfrak{sl}_2$, so we have an isomorphism $\rho\colon S^2(V_1)\to\mathfrak{sl}_2$. Let $\langle\,,\,\rangle\colon\Lambda^2(V_1)\to\mathsf{k}$

[1]Here we list only exceptional Lie superalgebras $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$ with $\mathfrak{g}_1\neq 0$.

be a nondegenerate $\mathfrak{sl}_2$-invariant skew-symmetric form on V_1. The Lie bracket $D(2,1;a)_1 \times D(2,1;a)_1 \to D(2,1;a)_0$ is defined by

$$[v_1 \otimes v_2 \otimes v_3, w_1 \otimes w_2 \otimes w_3] = \alpha\langle v_1, w_1\rangle\langle v_2, w_2\rangle\rho(v_3, w_3) \\ + \beta\langle v_1, w_1\rangle\rho(v_2, w_2)\langle v_3, w_3\rangle + \gamma\rho(v_1, w_1)\langle v_2, w_2\rangle\langle v_3, w_3\rangle$$

where α, β, and γ are some constants.

Check that $[x,[x,x]] = 0$ holds if $\alpha + \beta + \gamma = 0$.

Assume that $\alpha + \beta + \gamma = 0$. Since the algebra defined above is isomorphic to the one with the triple (α, β, γ) replaced by $(c\alpha, c\beta, c\gamma)$ for some constant c, each triple (α, β, γ) defines a point $a \in \mathbb{C}\mathrm{P}^1$. We denote by $D(2,1,a)$ the corresponding Lie superalgebra. It is easy to check that $D(2,1,a)$ is simple iff $a \in \mathbb{C}\mathrm{P}^1 \setminus \{0, 1, \infty\}$.

(2) The exceptional superalgebras $G(1|2)$ and $F(1|3)$[2] are particular cases of the following construction. Let $\mathfrak{g}_0 = \mathfrak{sl}_2 \oplus \mathfrak{k}$, where $\mathfrak{k}$ is a simple Lie algebra and $\mathfrak{g}_1 = V_1 \otimes V$, where V is a simple $\mathfrak{k}$-module. Assume that there exist a $\mathfrak{k}$-invariant symmetric form $b(\cdot,\cdot)$ on V. Then the adjoint $\mathfrak{k}$-module is a submodule in $\Lambda^2(V) \subset V \otimes V^*$, where V^* is identified with V by means of b. Hence there is a homomorphism $s : \Lambda^2(V) \to \mathfrak{k}$ of $\mathfrak{k}$-modules. Define a bracket $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ by

$$[v \otimes w, v' \otimes w'] = \langle v, v'\rangle s(w, w') + b(w, w')\rho(v, v')$$

for any $v, v' \in V_1$, $w, w' \in V$. If we are lucky and (3) holds, we get a simple Lie superalgebra.

To construct $G(1|2)$, let $\mathfrak{k}$ be of type G_2, and V be the unique 7-dimensional G_2-module.

To construct $F(1|3)$, let $\mathfrak{k} \simeq \mathfrak{o}(7)$ and V be the 8-dimensional spinor $\mathfrak{o}(7)$-module.

1.4.3 Classical but not Contragredient Superalgebras

In the case where $\dim V_0 = \dim V_1$, the Lie superalgebra $\mathfrak{gl}(V)$ has simple subalgebras which do not have analogues in the purely even case.

(1) Assume that $V = V_0 \oplus V_1$ and $\dim V_0 = \dim V_1 = n$. Define

$$\mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathfrak{psl}(n|n) \,\middle|\, \operatorname{tr} B = 0 \right\}.$$

The superalgebra $\mathfrak{q}(n)$ is simple if $n > 2$.

(2) Let $(\cdot,\cdot)$ be a nondegenerate odd symmetric form on V. Then $\mathfrak{p}(n)$ is the subalgebra of $\mathfrak{sl}(n|n)$ that preserves $(\cdot,\cdot)$. The matrix presentation of $\mathfrak{p}(n)$ is

$$\mathfrak{p}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sl}(n|n) \,\middle|\, B^t = B,\ C^t = -C,\ \operatorname{tr} A = 0 \right\}.$$

The superalgebra $\mathfrak{p}(n)$ is simple for $n > 2$.

[2] We modify Kac's original notation in order to avoid confusion with the usual Lie algebra F_4.

1.4.4 Cartan-Type Superalgebras

Let L be a $\mathbb{Z}$-graded Lie algebra satisfying the conditions

$$L = \bigoplus_{i=-k}^{\infty} L_i, \quad \dim L_i < \infty.$$

E. Cartan classified such simple infinite-dimensional Lie algebras. It turns out that the Cartan Lie algebras have finite-dimensional superanalogues. They are subsuperalgebras of $W(0|n)$.[3]

(1) For every $D = \sum_{i=1}^{n} f_i \frac{\partial}{\partial \xi_i}$, let

$$\operatorname{div} D = \sum_{i=1}^{n} (-1)^{p(f_i)} \frac{\partial f_i}{\partial \xi_i}.$$

Then

$$S(n) = \left\{ D \in W(0|n) | \operatorname{div} D = 0 \right\}$$

is a subalgebra of $W(0|n)$. It is simple for $n \geq 3$.

(2) If n is even, then

$$S'(n) = \left\{ D \in W(0|n) | D(1 + \xi_1 \cdots \xi_n) + \operatorname{div} D = 0 \right\}$$

is also a subalgebra of $W(0|n)$. It is simple for $n \geq 2$.

(3) Let

$$\tilde{H}(n) = \left\{ D \in W(0|n) | D = \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \xi_i}, \; f \in S(\xi_1, \ldots, \xi_n) \right\}.$$

Then $H(n) = [\tilde{H}(n), \tilde{H}(n)]$ is a subalgebra of codimension 1 in $\tilde{H}(n)$. It is simple for $n \geq 3$.

Exercise 1.3 *As in the Lie algebra case, there are isomorphisms in small dimensions. Establish the following isomorphisms*:

$$\mathfrak{osp}(2|2) \simeq \mathfrak{sl}(1|2) \simeq W(0|2),$$
$$\mathfrak{osp}(1|2) \simeq S'(2),$$
$$H(4) \simeq \mathfrak{psl}(2|2).$$

Exercise 1.4 *Show that* $D(2, 1; a)$ *is a one-parameter deformation of the Lie superalgebra* $\mathfrak{osp}(4|2)$.

[3] $W(0|n)$ is simple if $n \geq 2$.

One can see from the above list that most simple Lie superalgebras arise in a very natural way. One can also see that certain important properties of simple Lie algebras—semisimplicity of finite-dimensional modules, absence of deformations and central extensions, etc.—do not hold for simple Lie superalgebras. That makes the theory of Lie superalgebras and their representations interesting and difficult.

1.5 Algebraic Supergroups

The importance of Lie algebras is certainly related to the fact that they are an infinitesimal version of Lie groups. In fact, to a Lie algebra over $\mathbb{C}$ one can associate a complex Lie group. In many cases (for example, if the Lie algebra in question is simple) this group is algebraic. One can do the same for Lie superalgebras. In this section we will briefly discuss the notion of an algebraic supergroup. For a detailed treatment of this subject, we send the reader to the literature ([30], [31] for algebraic supergroups, [27], [28] for real Lie groups, and [42] for analytic supergroups).

There are two ways to define an affine algebraic group over k: as a commutative Hopf algebra or as a functor. While the second approach is more technical and requires the notion of representability, it has the advantage to be more geometric. Both approaches can be generalized to the supercase.

Let R be a commutative Noetherian Hopf superalgebra over k. Then for any supercommutative k-algebra S, the set $G(S)$ of nontrivial parity preserving homomorphisms $R \to S$ has a natural structure of a group with multiplication

$$g_1 g_2(r) = \sum_i g_1(r_i) g_2\big(r^i\big)$$

for any $g_1, g_2 \in G,\ r \in R$, and $\Delta(r) = \sum_i r_i \otimes r^i$, where Δ denotes the comultiplication in R. The associativity and the existence of identity and inverse are ensured by the Hopf superalgebra axioms.

Thus, a commutative Hopf superalgebra defines a functor from the category of supercommutative algebras to the category of groups. A general such functor G is called *representable* (in the category of commutative Hopf superalgebras) if it arises from a commutative Hopf superalgebra $R(G)$.

An *affine algebraic supergroup* G over k is a functor from the category of commutative k-superalgebras to the category of groups which is representable by a commutative Hopf superalgebra $R(G)$. If S is a commutative superalgebra, then the elements of $G(S)$ are called S-points of G. The algebra $R(G)$ is sometimes called the algebra of functions on G.

While in the usual theory of algebraic groups over $\mathbb{C}$ one can go pretty far without extending the base and only considering points over $S = \mathbb{C}$, in the case of supegroups one needs a base change right away since the set of $\mathbb{C}$-points of any supergroup is just a usual algebraic group G_0 with $R(G_0) = R(G)/(R_1(G))$.

If G is an algebraic supergroup, then the superalgebra of right-invariant derivations of $R(G)$,

$$\mathrm{Lie}(G) = \left\{ d \in \mathrm{Der}\, R(G) | (d \otimes 1) \circ \Delta = \Delta \circ d \right\},$$

is called the *Lie superalgebra of the algebraic supergroup G.*

Thus, we have a functor from the category of algebraic supergroups (or commutative Hopf superalgebras) to the category of Lie superalgebras.

Example Let S be a commutative superalgebra, and $S^{m|n}$ be a free S-module with m even and n odd generators. Define

$$GL(m|n, S) = \mathrm{Aut}_S(S^{m|n}).$$

Exercise 1.5 *Check that*

$$GL(m|n, S) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \begin{array}{ll} a_{i,j}, d_{i,j} \in S_0, & A, D \text{ are} \\ b_{i,j}, c_{i,j} \in S_1, & \text{invertible} \end{array} \right\}.$$

Show that $G = GL(m|n)$ is an algebraic supergroup. Check that $R(G)$ is the symmetric algebra $S(\mathfrak{gl}(m|n))$ localized by $\det A$ *and* $\det D$, *and that* $\mathrm{Lie}(G) = \mathfrak{gl}(m|n)$.

For any $X \in Gl(m|n, S)$, define

$$\mathrm{Ber}\, X = \frac{\det(A - BD^{-1}C)}{\det D}.$$

$\mathrm{Ber}\, X$ is called the *Berezinian* of X.

Exercise 1.6 *Check that* $\mathrm{Ber}(X)$ *is an invertible element of S_0 and* $\mathrm{Ber} : GL(m|n, S) \to S_0^*$ *is a homomorphism of the supergroups $GL(m|n) \to GL(1)$.*

Example The *special linear supergroup $SL(m|n)$* is the subsupergroup of $GL(m|n)$ defined by

$$SL(m|n, S) = \left\{ X \in GL(m|n, S) \mid \mathrm{Ber}\, X = 1 \right\}.$$

In other words, $R(SL(m|n)) = R(GL(m|n))/(\mathrm{Ber} - 1)$.

Exercise 1.7 *Show that* $\mathrm{Lie}(SL(m|n)) = \mathfrak{sl}(m|n)$.

2 Kac–Moody Lie Superalgebras

Kac–Moody Lie algebras are a very natural analogue of semisimple finite-dimensional Lie algebras. They have a deep theory and many applications. Only some finite-dimensional simple Lie superalgebras are of Kac–Moody type. Their structure and representation theory is currently better developed due to the possibility of adaptation methods from Lie algebra theory. In this section we review some results about Kac–Moody Lie superalgebras. From now on we assume our basic field k algebraically closed and of characteristic 0.

2.1 Superalgebras Defined by Cartan Matrices

Let $C = (c_{ij})$ be an $n \times n$-matrix. Let $I = \{1, 2, \dots, n\}$, and $p\colon I \to \mathbb{Z}_2$ be a map. (In what follows we refer to a map $p\colon S \to \mathbb{Z}_2$ as a *parity function*.) Denote by $\mathfrak{h}$ an even $(2n - \operatorname{rk} C)$-dimensional vector space. There exist (unique up to a linear transformation) $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ and $h_1, \dots, h_n \in \mathfrak{h}$ such that

$$\alpha_i(h_j) = c_{ji}.$$

Denote by $\widetilde{\mathfrak{g}}(C)$ the Lie superalgebra generated by $\mathfrak{h}$, $X_1, \dots, X_n$, and $Y_1, \dots, Y_n$, subject to the relations

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &= 0, \\ [X_i, Y_j] &= \delta_{ij} h_i, \\ [h, X_i] &= \alpha_i(h) X_i, \\ [h, Y_i] &= -\alpha_i(h) Y_i. \end{aligned} \tag{4}$$

Here we assume that $p(X_i) = p(Y_i) = p(i)$.

It can be shown that there exists a unique maximal ideal $\iota \in \widetilde{\mathfrak{g}}(C)$ such that $\iota \cap \mathfrak{h} = 0$ (see [25]).

The *contragredient Lie superalgebra* with Cartan matrix C is the quotient

$$\mathfrak{g}(C) = \widetilde{\mathfrak{g}}(C)/\iota.$$

Let D be a nondegenerate diagonal $n \times n$-matrix. It is easy to check that

$$\mathfrak{g}(DC) \simeq \mathfrak{g}(C).$$

Thus, without loss of generality we may assume the diagonal entries c_{ii} to be equal either 2 or 0. We call such matrices *normalized*.

Exercise 2.1 *Let $n = 1$. Show that*

(i) *If $C = (2)$ and $p(1) = 0$, then $\mathfrak{g}(C) \simeq \mathfrak{sl}(2)$.*
(ii) *If $C = (2)$ and $p(1) = 1$, then $\mathfrak{g}(C) \simeq \mathfrak{osp}(1|2)$.*
(iii) *If $C = (0)$ and $p(1) = 1$, then $\mathfrak{g}(C) \simeq \mathfrak{gl}(1|1)$.*

As we will see in the next section, $\mathfrak{g}(C) \simeq \mathfrak{g}(C')$ does not imply that $C = DC'$ for some diagonal matrix D.

Exercise 2.2 *Let*

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Show that

$$\mathfrak{gl}(2|2) = \mathfrak{g}(C)$$

with

$$h_1 = E_{11} - E_{22}, \qquad h_2 = E_{22} + E_{33}, \qquad h_3 = E_{33} - E_{44},$$

$$X_1 = E_{12}, \qquad X_2 = E_{23}, \qquad X_3 = E_{34},$$

$$Y_1 = E_{21}, \qquad Y_2 = E_{32}, \qquad Y_3 = E_{43},$$

where E_{ij} is the elementary matrix with the (i, j)th entry equal to 1 *and all other entries equal to* 0.

A contragredient Lie superalgebra $\mathfrak{g}(C)$ is called *quasisimple* if for every ideal $\mathfrak{j}$ of $\mathfrak{g}(C)$, either $\mathfrak{j} \subset \mathfrak{h}$ or $\mathfrak{j} + \mathfrak{h} = \mathfrak{g}(C)$. One can see from the previous example that quasisimplicity is not equivalent to simplicity. In the Lie algebra case, a typical example of a quasisimple but not simple contragredient Lie algebra is an affine Lie algebra.

Exercise 2.3 *Check that the quasisimplicity of $\mathfrak{g}(C)$ implies that C is indecomposable*, *i.e.*, *there is no proper $I' \subset I$ such that $c_{ij} \neq 0$ implies $i, j \in I'$ or $i, j \notin I'$.*

A matrix is *admissible* iff the corresponding normalized matrix C satisfies the following conditions:

(i) $c_{ii} = 0 \quad \Rightarrow \quad p(i) = 1;$

(ii) $c_{ii} = 2 \quad \Rightarrow \quad \begin{cases} c_{ij} \in \mathbb{Z}_{\leq 0}, & p(i) = 0, \\ c_{ij} \in 2\mathbb{Z}_{\leq 0}, & p(i) = 1. \end{cases}$

Recall that a linear operator $T \in \mathrm{End}_{\Bbbk}(V)$ is called *locally nilpotent* if for any $v \in V$, there exists $n(v) \in \mathbb{Z}_{>0}$ such that $T^{n(v)} v = 0$. The following lemma partially justifies the definition of an admissible matrix.

Lemma 2.4 *If a matrix C is admissible*, *then* ad_{X_i} *and* ad_{Y_i} *are locally nilpotent in $\mathfrak{g}(C)$ for all $i \in I$. Conversely*, *if $\mathfrak{g}(C)$ is quasisimple and* ad_{X_i}, ad_{Y_i} *are locally nilpotent for all $i \in I$*, *then C is admissible.*

Proof If $c_{ii} = 0$, then $[X_i, X_i] = 2X_i^2 = 0$ by Exercise 2.1. Hence $\mathrm{ad}^2_{X_i} = 0$. If $c_{ii} = 2$, one can check by a direct computation that

$$\left[Y_k, \mathrm{ad}_{X_i}^{1-c_{ij}} X_j\right] = 0$$

for any k and $j \neq i$. Therefore $\mathrm{ad}_{X_i}^{1-c_{ij}} X_j$ generates an ideal in $\mathfrak{g}(C)$ which belongs to the subalgebra generated by $X_j, j \in I$. Such an ideal is zero, hence $\mathrm{ad}_{X_i}^{1-c_{ij}} X_j = 0$. Now it is trivial to check that ad_{X_i} acts nilpotently on all generators, and hence, by the "super"-Leibniz identity, on the entire algebra $\mathfrak{g}(C)$. The proof for Y_i is similar.

Now let $\mathfrak{g}(C)$ be quasisimple. Assume that $\mathrm{ad}_{X_i}, \mathrm{ad}_{Y_i}$ are locally nilpotent. First, consider the case where $p(i) = 0$. If $c_{ii} = 0$, then X_i, Y_i, h_i generate a Heisenberg Lie algebra, and by Engel's theorem ad_{h_i} is nilpotent. On the other hand, ad_{h_i} is semisimple, hence h_i belongs to the center of $\mathfrak{g}(C)$. Therefore h_i and X_i generate an ideal in $\mathfrak{g}(C)$ which does not contain any Y_j. This contradicts the assumption of quasisimplicity. Hence (i) holds.

If $c_{ii} = 2$, then X_i, Y_i, h_i generate an $\mathfrak{sl}(2)$-subalgebra, and $\mathfrak{g}(C)$ is a direct sum of finite-dimensional $\mathfrak{sl}(2)$-modules. For any $j \neq i$, X_j is the lowest weight vector, hence must have a nonpositive integral weight c_{ij}. We obtain (ii) in the case where $p(i) = 0$.

Now let $p(i) = 1$. If $c_{ii} = 0$, there is nothing to prove. If $c_{ii} = 2$, then by Exercise 2.1, $\{\frac{[X_i,X_i]}{2}, \frac{h_i}{2}, \frac{[Y_i,Y_i]}{2}\}$ form an $\mathfrak{sl}(2)$-triple, and we can repeat the previous argument. The proof is complete. □

We call a matrix C and the Lie superalgebra $\mathfrak{g}(C)$ *regular* if $c_{ij} = 0$ implies $c_{ji} = 0$ for all $i, j \in I$. In principle, quasisimplicity does not imply regularity as we will see in Sect. 3.4.

2.1.1 Root Decomposition

As in the Lie algebra case, $\mathfrak{g} = \mathfrak{g}(C)$ has the *root decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

for some subset $\Delta \subset \mathfrak{h}^* \setminus \{0\}$, where

$$\mathfrak{g}_\alpha = \left\{x \in \mathfrak{g} | [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\right\}.$$

The set Δ is called the set of *roots*. By definition $\alpha_1, \ldots, \alpha_n$ are roots, and every root $\alpha \in \Delta$ can be written uniquely as $\alpha = \sum_i n_i \alpha_i$, where all n_i are either nonnegative or nonpositive integers. If $n_i \geq 0$ (resp., $n_i \leq 0$), then α is called positive (resp., negative). By $\Delta^{\pm}$ we denote the set of positive (resp., negative roots). The abelian group $Q \subset \mathfrak{h}^*$ generated by Δ is called the *root lattice*. It is easy to see that there is a unique additive parity function $p : Q \to \mathbb{Z}_2$ such that $p(\alpha_i) = p(i)$. The root space $\mathfrak{g}_\alpha$ is purely even or purely odd depending on the parity of α.

2.2 Odd Reflections and Regular Kac–Moody Superalgebras

In the theory of Kac–Moody Lie algebras an important role is played by the Weyl group. The generalization of this notion to the supercase is quite interesting.

For each simple root α_i, there is a way to construct a new system of generators of $\mathfrak{g}(C)$. If $c_{ii}=2$, this leads to an automorphism of $\mathfrak{g}(C)$, but if $c_{ii}=0$, we obtain a new Cartan matrix.

Let α_i be a simple root such that $c_{ii}=2$. The map

$$r_{\alpha_i}:\Delta\to\Delta: r_{\alpha_i}(\alpha)=\alpha-\alpha(h_i)\alpha_i$$

is called an *even reflection*. Assume that C is regular admissible. It is an easy exercise to check that the map

$$X_\alpha\mapsto X_{r_{\alpha_i}(\alpha)},\qquad Y_\alpha\mapsto Y_{r_{\alpha_i}(\alpha)},\qquad h_\alpha\mapsto h_{r_{\alpha_i}(\alpha)}=[X_{r_{\alpha_i}(\alpha)},Y_{r\alpha_i(\alpha)}],$$

can be extended to an automorphism of $\mathfrak{g}(C)$.

For every $k\in I$ such that $c_{kk}=0$, set

$$r_{\alpha_k}(\alpha_j)=\begin{cases}\alpha_j+\alpha_k, & \alpha_j(h_k)\neq 0,\ j\neq k,\\ \alpha_j, & \alpha_j(h_k)=0,\ j\neq k,\\ -\alpha_k, & j=k,\end{cases}$$

and

$$X_j'=\begin{cases}[X_k,X_j] & \text{if } c_{kj}\neq 0,\ j\neq k,\\ X_j & \text{if } c_{kj}=0,\ j\neq k,\\ Y_k & \text{if } j=k,\end{cases}$$

$$Y_j'=\begin{cases}[Y_k,Y_j] & \text{if } c_{kj}\neq 0,\ j\neq k,\\ Y_j & \text{if } c_{kj}=0,\ j\neq k,\\ X_k & \text{if } j=k.\end{cases}$$

A linearly independent set Σ of roots of $\mathfrak{g}(C)$ is called a *base* if for every $\beta\in\Sigma$, there exist $X_\beta\in\mathfrak{g}_\beta$ and $Y_\beta\in\mathfrak{g}_{-\beta}$ such that X_β, Y_β for $\beta\in\Sigma$ (together with $\mathfrak{h}$) generate $\mathfrak{g}(C)$, and

$$[X_\beta,Y_\gamma]=0$$

for any $\beta,\gamma\in\Sigma$, $\beta\neq\gamma$. If we put $h_\beta=[X_\beta,Y_\beta]$, then X_β, Y_β, and h_β satisfy the following relations:

$$[h,X_\beta]=\beta(h)X_\beta,\qquad [h,Y_\beta]=-\beta(h)Y_\beta,\qquad [X_\beta,Y_\gamma]=\delta_{\beta\gamma}h_\beta.$$

Lemma 2.5 *Let C be regular admissible, $k\in I$, $\alpha_j'=r_{\alpha_k}(\alpha_j)$, X_j', Y_j' be non-zero elements of $\mathfrak{g}_{r_{\alpha_k}(\alpha_j)}$ and $\mathfrak{g}_{-r_{\alpha_k}(\alpha_j)}$ respectively, and $h_j':=[X_j',Y_j']$. Then*

$X'_1, \dots, X'_n, Y'_1, \dots, Y'_n$, and $\mathfrak{h}$ generate $\mathfrak{g}(C)$, and $\{\alpha'_1, \dots, \alpha'_n\}$ is a base of $\mathfrak{g}(C)$. Moreover, $\mathfrak{g}(C) \simeq \mathfrak{g}(C')$, where C' is the Cartan matrix with entries $c'_{ij} = \alpha'_j(h'_i)$.

Proof Straightforward computations, see [39]. □

We say that the matrix C' is obtained from the matrix C by the odd reflection r_{α_i}. Whenever $k\alpha$ is a root, we say that $r_{k\alpha} = r_\alpha$. As follows from Lemma 2.5, if Σ is a base and $\alpha \in \Sigma$, then $r_\alpha(\Sigma)$ is again a base. On the other hand, the action of an odd reflection r_α cannot be extended to the entire set of roots. Thus, we cannot construct a group which is generated by all reflections (even and odd). But it is possible to define a groupoid generated by all reflections (see [39]).

Exercise 2.6 *If r_α is an even reflection, then for any reflection r_β, the following identity holds:*

$$r_\alpha r_\beta r_\alpha = r_{r_\alpha(\beta)}.$$

Example The Lie algebra $\mathfrak{sl}(1|2)$ has the following Cartan matrices connected by odd reflections:

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \underset{r_1}{\overset{r_1}{\rightleftarrows}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{r_2}{\overset{r_2}{\rightleftarrows}} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

A contragradient Lie superalgebra $\mathfrak{g}(C)$ is called a *regular Kac–Moody superalgebra* if every C', obtained from C by several applications of odd reflections, is admissible and regular.

Theorem 1 [39] *Let $\mathfrak{g}(C)$ be a regular Kac–Moody superalgebra. Then every base can be obtained from a given one by applying even and odd reflections and an automorphism $\varphi\colon X_i \leftrightarrow Y_i$ (in the infinite-dimensional case).*

Proof A general proof I know is based on the classification of regular Kac–Moody superalgebras. Here I only sketch a proof for a finite-dimensional $\mathfrak{g}(C)$. Let Π and Σ be two bases, and $\Delta^+(\Pi)$, $\Delta^+(\Sigma)$ be the corresponding sets of positive roots. If $\Pi \neq \Sigma$, there exists a root $\alpha \in \Pi$ such that $\alpha \notin \Sigma$. Let $\Pi' = r_\alpha(\Pi)$. Then

$$\Delta^+(\Pi') = \Delta^+(\Pi) \cup \{-\alpha\} \setminus \{\alpha\}$$

or

$$\Delta^+(\Pi') = \Delta^+(\Pi) \cup \{-\alpha, -2\alpha\} \setminus \{\alpha, 2\alpha\},$$

depending on whether 2α is a root. In both cases, the cardinality of $\Delta^+(\Pi') \setminus \Delta^+(\Sigma)$ is less than the cardinality of $\Delta^+(\Pi) \setminus \Delta^+(\Sigma)$. Since Δ is finite, one can finish the proof by induction on the cardinality of $\Delta^+(\Pi) \setminus \Delta^+(\Sigma)$. □

A root α is called *real* if α or $\alpha/2$ is a root of some base. The Weyl group of a regular Kac–Moody superalgebra is the subgroup of automorphisms of $\mathfrak{g}$ generated by r_β for all even real roots β.

The following statement is a direct consequence of Lemma 2.4.

Lemma 2.7 *If* $\alpha \in \Delta$ *is real, then* $\dim \mathfrak{g}_\alpha = (1|0)$ *or* $(0|1)$, *and* ad_X *is locally nilpotent for any* $X \in \mathfrak{g}_\alpha$.

Exercise 2.8 *Show that a regular Kac–Moody superalgebra* $\mathfrak{g}(C)$ *is finite dimensional iff all roots are real.*

Example The Lie superalgebra $D(2,1;a)$ has the Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & a \\ 0 & -1 & 2 \end{pmatrix}.$$

The following diagram shows other Cartan matrices obtained from C by odd reflections:

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & a \\ 0 & -1 & 2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 1 & -1-a \\ 1 & 0 & a \\ 1-\frac{1}{a} & 1 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & -1-a \\ 0 & -1 & 2 \end{pmatrix}$$

$$\updownarrow$$

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & -1-\frac{1}{a} \\ 0 & -1 & 2 \end{pmatrix}$$

Note that this diagram implies the following isomorphisms:

$$D(2,1;a) \simeq D(2,1;-1-a) \simeq D\left(2,1;-1-\frac{1}{a}\right).$$

One can see that the group S_3 generated by the maps $a \to \frac{1}{a}$ and $a \to -1-a$ acts on the space of parameters, and the points of an orbit of this action correspond to isomorphic superalgebras. Hence the moduli space of $D(2,1;a)$ is $\mathbb{C}\mathrm{P}^1 \setminus \{0,-1,\infty\}$ modulo the above S_3-action. The points $0,-1,\infty$ correspond to isomorphic non-regular Kac–Moody superalgebras.

2.3 Symmetrizable Kac–Moody Superalgebras and Affine Superalgebras

A matrix C is called *symmetrizable* if there is an invertible diagonal matrix D such that DC is symmetric. Obviously a symmetrizable matrix is regular, and the property to be symmetrizable is preserved by odd reflections. We say that $\mathfrak{g}(C)$ is symmetrizable if C is symmetrizable.

Lemma 2.1 *If $\mathfrak{g}(C)$ is symmetrizable, then there is a nondegenerate even symmetric invariant form $(\cdot,\cdot)$ on $\mathfrak{g}(C)$*

The proof is exactly the same as in the Lie algebra case (see [25]) and can be found in [29].

It is easy to see that the restriction of $(\cdot,\cdot)$ on $\mathfrak{h}$ is nondegenerate, $(\mathfrak{g}_\alpha,\mathfrak{g}_\beta)=0$ if $\alpha\neq-\beta$, and $(\cdot,\cdot)$ defines the nondegenerate pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. By η we denote the isomorphism $\mathfrak{h}\to\mathfrak{h}^*$ induced by $(\cdot,\cdot)$. By a slight abuse of notation we denote the corresponding form on $\mathfrak{h}^*$ by the same symbol.

Exercise 2.9 *Show that*

$$[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]=\mathsf{k}\eta^{-1}(\alpha)$$

for any $\alpha\in\Delta$.

All finite-dimensional contragredient simple Lie superalgebras are symmetrizable, as one can see from the classification. Another natural class of symmetrizable superalgebras is the class of affine superalgebras. Here is the definition.

Let $\mathfrak{s}$ be a simple contragredient Lie superalgebra, $\mathfrak{s}\neq\mathfrak{psl}(n|n)$, and $(\cdot,\cdot)$ be an invariant symmetric even form on $\mathfrak{s}$. The *affine superalgebra* $\mathfrak{s}^{(1)}$ is the infinite-dimensional vector superspace

$$\mathfrak{s}\otimes\mathsf{k}\big[t,t^{-1}\big]\oplus\mathsf{k}D\oplus\mathsf{k}K$$

with Lie bracket defined by

$$\big[x\otimes t^k,y\otimes t^l\big]=[x,y]\otimes t^{k+l}+k\delta_{k,-l}(x,y)K,$$
$$[K,D]=\big[K,x\otimes t^l\big]=0,\qquad \big[D,x\otimes t^l\big]=lx\otimes t^l$$

for any $x,y\in\mathfrak{s}$ and $k,l\in\mathbb{Z}$.

Exercise 2.10 *Let $X_1,\dots,X_n,Y_1,\dots,Y_n$ be generators of $\mathfrak{s}$ satisfying (4). There exist (unique up to proportionality) $x_0,y_0\in\mathfrak{s}$ such that $[Y_i,x_0]=[X_i,y_0]=0$ for all $i>0$. Set $X_0=x_0\otimes t^{-1}$ and $Y_0=y_0\otimes t$. Check that $X_0,X_1,\dots,X_n,Y_0,Y_1,\dots,Y_n$ satisfy (4) and, together with K and D, generate $\mathfrak{s}^{(1)}$.*

The reason why we exclude the Lie superalgebra $\mathfrak{s}=\mathfrak{psl}(n|n)$ is related to the fact that its Cartan matrix does not have maximal rank. In particular, the corresponding Kac–Moody superalgebra is not simple but quasisimple and is isomorphic to $\mathfrak{gl}(n|n)$, see Example 2.2. The affinization $\mathfrak{psl}(n|n)^{(1)}$ is the infinite-dimensional vector superspace

$$\mathfrak{s}\otimes\mathsf{k}\big[t,t^{-1}\big]\oplus\mathsf{k}D\oplus\mathsf{k}K\oplus\mathsf{k}D'\oplus\mathsf{k}K'$$

with Lie bracket defined by

$$\begin{gathered}
\left[x \otimes t^k, y \otimes t^l\right] = [x, y] \otimes t^{k+l} + k\delta_{k,-l}(x, y)K + p(x)p(y)\,\mathrm{str}(xy)K',\\
[K, D] = [K', D] = [K, D'] = [D, D'] = \left[K, x \otimes t^l\right] = 0,\\
\left[D, x \otimes t^l\right] = lx \otimes t^l, \qquad \left[D', x \otimes t^l\right] = p(x)x \otimes t^l
\end{gathered}$$

for any $x, y \in \mathfrak{s} = \mathfrak{psl}(n|n)$, $n > 1$, $k, l \in \mathbb{Z}$.

Now let ϕ be an automorphism of $\mathfrak{s}$ of finite order m which preserves the form $(\cdot, \cdot)$. Let ε be the mth primitive root of 1. We extend ϕ to an automorphism $\bar{\phi}$ of $\mathfrak{s}^{(1)}$ by putting

$$\begin{aligned}
\bar{\phi}(K) &= K, & \bar{\phi}(D) &= D,\\
\bar{\phi}(K') &= K', & \bar{\phi}(D') &= D'
\end{aligned}$$

for $\mathfrak{s} \simeq \mathfrak{psl}(n|n)$) and

$$\bar{\phi}\left(x \otimes t^k\right) = \varepsilon^{-k}\phi(x) \otimes t^k$$

for all $x \in \mathfrak{s}$, $k \in \mathbb{Z}$. The subsuperalgebra of the fixed points of $\bar{\phi}$ is called a *twisted affine superalgebra*. We denote it by $\mathfrak{s}^{\phi}$. It is possible to show that if two automorphisms ϕ and ψ belong to the same connected component of the group of automorphisms of $\mathfrak{s}$, then $\mathfrak{s}^{\phi}$ and $\mathfrak{s}^{\psi}$ are isomorphic. Usually the notation $\mathfrak{s}^{(m)}$ is used if the choice of ϕ is clear.

It can be shown (see, for example, [29]) that all affine and twisted affine superalgebras are symmetrizable Kac–Moody superalgebras.

2.4 Classification Results

Define a $\mathbb{Z}$-grading $\mathfrak{g}(C) = \bigoplus \mathfrak{g}_n$ by setting $\deg X_i = 1$, $\deg Y_i = -1$, and $\deg \mathfrak{h} = 0$.

A contragredient Lie (super)algebra $\mathfrak{g}(C)$ has *finite growth* if $\dim \mathfrak{g}_n < P(n)$ for some polynomial $P(z)$.

Exercise 2.11 *Check that the property to have finite growth does not depend on the choice of a base in* $\mathfrak{g}(C)$.

In the Lie algebra case it was proved by V. Kac that all finite growth Kac–Moody Lie algebras are either finite dimensional or (twisted) affine. He also proved that the same is true for Lie superalgebras in the case where Cartan matrices do not have zero entries on the diagonal, [21].

In the supercase the similar result is true for symmetrizable contragredient Lie superalgebras.

Theorem 2 [29] *Any contragredient Lie superalgebra of finite growth with an indecomposable symmetrizable Cartan matrix is either finite dimensional or* (*twisted*) *affine*.

The following theorem justifies our definition of Kac–Moody Lie superalgebra. The proof is not difficult but quite long. It is based on the analysis of rank 2 and rank 3 cases.

Theorem 3 [14] *If* $\mathfrak{g}(C)$ *is quasisimple regular and has finite growth*, *then* $\mathfrak{g}(C)$ *is a regular Kac–Moody Lie superalgebra*.

All regular quasisimple Kac–Moody superalgebras which have at least one zero entry on the diagonal were classified by Hoyt [15]. Based on this classification, a complete classification of contragedient quasisimple Lie superalgebras of finite growth was obtained. Before formulating this result, we give two examples of nonsymmetrizable Kac–Moody Lie superalgebras.

(1) Let φ be an involutive automorphism of the Lie superalgebra $\mathfrak{q}(n)$ given by $\varphi(x) = (-1)^{p(x)}x$. Although the Lie superalgebras $\mathfrak{q}(n)$ and $\mathfrak{q}(n)^{(1)}$ are not contragradient, twisting by φ yields a nonsymmetrizable regular Kac–Moody superalgebra, which we denote by $\mathfrak{q}(n)^{(2)}$. As a vector space, $\mathfrak{q}(n)^{(2)}$ is isomorphic to

$$\mathfrak{sl}(n) \otimes \mathsf{k}\big[t, t^{-1}\big] \oplus \mathsf{k}K \oplus \mathsf{k}D.$$

The commutator is given by the formula

$$\big[x \otimes t^k, y \otimes t^l\big] = \big(xy - (-1)^{kl} yx\big) \otimes t^{k+l} + \delta_{k,-l}\big(1 - (-1)^k\big) \operatorname{tr}(xy) K,$$
$$[K, D] = \big[K, x \otimes t^l\big] = 0, \qquad \big[D, x \otimes t^l\big] = x \otimes t^l,$$

for all $x, y \in \mathfrak{q}(n)$ and $k, l \in \mathbb{Z}$. One of possible Cartan matrices for the case $n = 3$ is

$$C = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

(2) The Lie superalgebra $S(1, 2; b)$ is a one-parameter deformation of the affine superalgebra $\mathfrak{sl}(1|2)^{(1)}$ with Cartan matrix

$$C = \begin{pmatrix} 0 & b & 1-b \\ -b & 0 & 1+b \\ -1 & -1 & 2 \end{pmatrix}$$

and parity $(1, 0, 0)$. We see that C is regular if $b \neq \pm 1$. The odd reflections r_1 and r_2 induce the isomorphisms $S(1, 2; b) \simeq S(1, 2; 1-b) \simeq S(1, 2; -b)$. Hence $S(1, 2; b)$

is a regular Kac–Moody superalgebra if $b \notin \mathbb{Z}$. When $b \in \mathbb{Z}$, $S(1,2;b)$ is quasisimple but not regular. If we make an odd reflection when $b = \pm 1$, we obtain a Cartan matrix of $\mathfrak{gl}(2|2)$ which is a subalgebra in $S(1,2;b)$.

To see that $S(1,2;b)$ is a deformation of $\mathfrak{sl}(1|2)^{(1)}$, renormalize C

$$\begin{pmatrix} 0 & 1 & -1-a \\ 1 & 0 & -1+a \\ -1 & -1 & 2 \end{pmatrix}$$

with $a = \frac{1}{b}$. When $a = 0$, we obtain a Cartan matrix of $\mathfrak{sl}(1|2)^{(1)}$.

For a realization of $S(1,2;b)$ as a subsuperalgebra of the Lie superalgebra $\operatorname{Der} \mathsf{k}[t, t^{-}1] \otimes \Lambda(\xi_1, \xi_2)$, see [39]. Quite remarkably, $S(1,2;b)$ is a conformal superalgebra in the sense of Kac and Van de Leur [24], in particular, it has a subalgebra isomorphic to the Virasoro algebra.

Theorem 4 [14] *A nonsymmetrizable quasisimple contragredient Lie superalgebra of finite growth is isomorphic to* $\mathfrak{q}(n)^{(2)}$ *or* $S(1,2;b)$.

3 Representation Theory of Kac–Moody Superalgebras

3.1 General Remarks About Modules over Lie Superalgebras

Let $\mathfrak{g}$ be a Lie superalgebra. A vector superspace M is a $\mathfrak{g}$-module if there is a parity preserving linear map $\mathfrak{g} \otimes M \to M$ satisfying

$$xym - (-1)^{p(x)p(y)} yxm = [x,y]m.$$

Equivalently, one can define a module as a homomorphism of Lie superalgebras $\mathfrak{g} \to \operatorname{End}_{\mathsf{k}}(M)$.

The notions of simple (irreducible) module, indecomposable module, direct sum, and tensor product have obvious superanalogues. We leave them to the reader.

The Schur lemma should be slightly modified in the supercase as was pointed out in [19].

Lemma 3.1 *Let M be a simple finite-dimensional $\mathfrak{g}$-module. Then any nonzero homogeneous $\phi \in \operatorname{End}_{\mathfrak{g}}(M)$ is invertible. If k is algebraically closed, then* $\dim \operatorname{End}_{\mathfrak{g}}(M) = (1|0)$ *or* $(1|1)$.

Define the *universal enveloping (super) algebra* $U(\mathfrak{g})$ as the quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal generated by $xy - (-1)^{p(x)p(y)} yx$ for all homogeneous $x, y \in \mathfrak{g}$. As in the Lie algebra case, the Poincaré–Birkhoff–Witt theorem is true, i.e., the associated graded algebra of $U(\mathfrak{g})$ (with respect to the natural filtration) is isomorphic to the symmetric superalgebra $S(\mathfrak{g})$. Furthermore, the category of $\mathfrak{g}$-modules is equivalent to the category of $U(\mathfrak{g})$-modules.

If $\mathfrak{k} \subset \mathfrak{g}$ is a Lie subsuperalgebra and M is a $\mathfrak{k}$-module, we define the induced and coinduced modules as $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} M$ and $\mathrm{Hom}_{U(\mathfrak{k})}(U(\mathfrak{g}), M)$, respectively. If $\mathfrak{g}$ is finite dimensional, $\mathfrak{g}_0 \subset \mathfrak{k}$, and M is finite dimensional, then both induced and coinduced modules are finite dimensional.

Exercise 3.1 *Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra and M be a $\mathfrak{g}_0$-module. Then*

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M \simeq \mathrm{Hom}_{U(\mathfrak{g}_0)}\big(U(\mathfrak{g}), M \otimes S^p(\mathfrak{g}_1)\big),$$

where $p = \dim \mathfrak{g}_1$.

3.2 Weight Modules and Integrable Modules

Let $\mathfrak{g} = \mathfrak{g}(C)$ be a regular Kac–Moody superalgebra. A $\mathfrak{g}$-module M is called a *weight module* if $\mathfrak{h}$ acts semisimply on M, i. e. M admits the following decomposition:

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu, \quad M_\mu = \big\{m \in M | hm = \mu(h)m, \ \forall h \in \mathfrak{h}\big\},$$

and $\dim M_\mu < \infty$ for all $\mu \in \mathfrak{h}^*$.

The set

$$P(M) = \{\mu \in \mathfrak{h}^* | M_\mu \neq 0\}$$

is called the *set of weights* of M.

Exercise 3.2 *A submodule and a quotient of a weight module is a weight module.*

The character of a weight module M is by definition the formal expression

$$\mathrm{ch}\, M = \sum_{\mu \in P(M)} \big(\dim(M_\mu)_0 + \dim(M_\mu)_1\big) e^\mu.$$

Exercise 3.3 *If M and N are weight modules, then*

$$\mathrm{ch}(M \oplus N) = \mathrm{ch}\, M + \mathrm{ch}\, N.$$

If $M \otimes N$ is a weight module, then

$$\mathrm{ch}(M \otimes N) = \mathrm{ch}\, M\, \mathrm{ch}\, N.$$

Exercise 3.4 *If M is a cyclic weight module, i.e. M is generated by a homogeneous vector, then for any $\mu \in P(M)$ either* $\dim(M_\mu)_0 = 0$ *or* $\dim(M_\mu)_1 = 0$. *Therefore for a cyclic weight module the character contains all information about dimensions of M_μ up to change of parity.*

A weight module M is called *integrable* if $\mathfrak{g}_\beta$ acts locally nilpotently on M for every real root β of $\mathfrak{g}$. For example the adjoint module is integrable.

Exercise 3.5 *If $\mathfrak{g}$ is finite dimensional, then any simple finite-dimensional $\mathfrak{g}$-module is an integrable weight module.*

Exercise 3.6 *If M is an integrable weight module, then for any even real positive root β of $\mathfrak{g}$, the operator*

$$r_\beta = \exp X_\beta \exp(-Y_\beta) \exp X_\beta$$

is well defined on M. For any $\mu \in P(M)$, we have $r_\beta(M_\mu) = M_{r_\beta(\mu)}$. In particular, $\operatorname{ch} M$ *is W-invariant.*

3.3 Highest Weight Modules

Fix a base Π of $\mathfrak{g}(C)$ and let Δ^+ (resp., Δ^-) denote the corresponding set of positive (resp., negative) roots. Then $\mathfrak{g}(C)$ has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where

$$\mathfrak{n}^\pm = \bigoplus_{\alpha\in\Delta^\pm} \mathfrak{g}_\alpha.$$

For each $\lambda \in \mathfrak{h}^*$, define the *Verma module* $M(\lambda)$ with *highest weight* λ as the induced module

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_\lambda,$$

where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, and C_λ is the one-dimensional module with an even generator v such that $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$ and $X_\alpha v = 0$ for all $\alpha \in \Pi$.

Proposition 3.2 *$M(\lambda)$ has a unique proper maximal submodule, and the quotient by this submodule is a simple $\mathfrak{g}(C)$-module which we denote by $L(\lambda)$.*

If $\mathfrak{g}(C)$ is finite dimensional, then every finite-dimensional simple $\mathfrak{g}(C)$-module is isomorphic to either $L(\lambda)$ or $\pi(L(\lambda))$.

The proof of the proposition is standard and can be found, for example, in [20].

If $\mathfrak{g}$ is symmetrizable, then as in the Lie algebra case one can define the Casimir operator (see [25]). Let us recall its construction. For every $\alpha \in \Delta^+$, choose a basis $\{e_\alpha^1, \ldots, e_\alpha^{m(\alpha)}\}$ in $\mathfrak{g}_\alpha$ and the dual basis $\{f_\alpha^1, \ldots, f_\alpha^{m(\alpha)}\}$ in $\mathfrak{g}_{-\alpha}$ with respect to the invariant symmetric form on $\mathfrak{g}$. Let $\{u_i\}$ be some orthonormal basis in $\mathfrak{h}$. There exists $\rho \in \mathfrak{h}^*$ satisfying the condition $\rho(h_\alpha) = \frac{1}{2}\alpha(h_\alpha)$ for any $\alpha \in \Pi$. Set

$$\Omega = 2\eta^{-1}(\rho) + \sum_{i=1}^{\dim \mathfrak{h}} u_i^2 + 2 \sum_{\alpha \in \Delta^+} (-1)^{p(\alpha)} \sum_{j=1}^{m(\alpha)} f_\alpha^j e_\alpha^j.$$

It is clear that Ω is a well-defined operator on $M(\lambda)$. By a direct computation one can check that $\Omega \in \mathrm{End}_{\mathfrak{g}}(M(\lambda))$ and that $\Omega v = (\lambda + 2\rho, \lambda)v$. Hence $\Omega = (\lambda + 2\rho, \lambda)\,\mathrm{id}$ on $M(\lambda)$. The existence of the Casimir operator has many consequences. Here is one of them (see [22]).

Lemma 3.3 *Assume that* $\mathfrak{g}$ *is symmetrizable and* $\lambda \in \mathfrak{h}^*$. *If* $2(\lambda + \rho, \alpha) \neq m(\alpha, \alpha)$ *for any* $\alpha \in \Delta^+$ *and* $m \in \mathbb{N}$, *then the Verma module* $M(\lambda)$ *is irreducible.*

Proof Let

$$Q^+ = \left\{ \sum_{\alpha \in \Pi} n_\alpha \alpha | n_\alpha \in \mathbb{Z}_{\geq 0} \right\},$$

and for any $\gamma = \sum_{\alpha \in \Pi} n_\alpha \alpha \in Q^+$, set $|\gamma| = \sum_{\alpha \in \Pi} n_\alpha$. It is easy see that $P(M(\lambda)) = \lambda - Q^+$.

If $M(\lambda)$ is reducible, then it has a nontrivial proper submodule $N(\lambda)$. Since $N(\lambda)$ is proper, $\lambda \notin P(N(\lambda))$. Pick $\mu = \lambda - \gamma \in P(N(\lambda))$ with minimal $|\gamma|$. Then $X_\alpha w = 0$ for any $w \in N(\lambda)_\mu$ and $\alpha \in \Pi$. A simple calculation shows that $\Omega w = (\mu + 2\rho, \mu)w$. Therefore we have

$$(\mu + 2\rho, \mu) = (\lambda + 2\rho, \lambda),$$

which implies

$$2(\lambda + \rho, \gamma) = (\gamma, \gamma). \tag{5}$$

For every $\gamma \in Q^+$, set

$$S(\gamma) = \left\{ \lambda \in \mathfrak{h}^* | \lambda - \gamma \in P\big(N(\lambda)\big) \right\}.$$

It is clear that $S(\gamma)$ is Zariski closed in $\mathfrak{h}^*$. To make this statement precise, we introduce the Shapovalov form. For any $x \in U(\mathfrak{n}^+))_\gamma$ and $y \in U(\mathfrak{n}^-))_{-\gamma}$,

$$XYv = s_{X,Y}(\lambda)v$$

for some polynomial $s_{X,Y}(\lambda)$. This polynomial defines a bilinear form

$$s_\gamma : U(\mathfrak{n}^+)_\gamma \times U(\mathfrak{n}^-)_{-\gamma} \to S(\mathfrak{h}^*),$$

which is called the Shapovalov form. Denote its determinant by $d_\gamma(\lambda)$. It is clear that $\lambda \in S(\gamma)$ iff $d_\gamma(\lambda) = 0$. It is not hard to show using Exercise 2.9 that the term of maximal degree of $d_\gamma(\lambda)$ is proportional to a product $\prod_{\beta \in F} (\lambda, \beta)$ for some $F \subset \Delta$. On the other hand, if λ and γ satisfy the assumption of the previous paragraph, then

(5) implies that if $d_\gamma(\lambda) = 0$, then $2(\lambda+\rho,\gamma) - (\gamma,\gamma) = 0$. Therefore $d_\gamma(\lambda)$ is proportional to some power of $2(\lambda+\rho,\gamma) - (\gamma,\gamma)$. From the statement about the leading term of $d_\gamma(\lambda)$ we obtain that $\gamma = m\beta$ for some $\beta \in \Delta^+$ and $m \in \mathbb{N}$.

Thus, if $M(\lambda)$ is reducible, one can find $\beta \in \Delta^+$ and $m \in \mathbb{N}$ such that

$$2(\lambda+\rho,\beta) = m(\beta,\beta).$$

Hence the statement. □

A general formula for the Shapovalov determinant is obtained in [11].

As far as we know, it is an open question whether for any (nonsymmetrizable) Kac–Moody Lie algebra, a generic Verma module is irreducible. On the other hand, we know that the answer is negative for Lie superalgebras. If $\mathfrak{g} = \mathfrak{q}(n)^{(2)}$, every Verma module is reducible (see [12]).

3.4 Integrable Highest Weight Modules

We call a weight λ integral dominant if the highest weight module $L(\lambda)$ is integrable.

Exercise 3.7 *If $\mathfrak{g}$ is finite dimensional, then λ is integral dominant iff* $\dim L(\lambda) < \infty$.

If $\mathfrak{g}$ is a Kac–Moody Lie algebra, then λ is integral dominant iff $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$ for all simple roots α [25].

If $\mathfrak{g}$ is a Lie superalgebra, the situation is somewhat more complicated.

Lemma 3.4 *Let $\mathfrak{g}$ be a regular Kac–Moody Lie superalgebra with a fixed base Π. Then λ is dominant integral if and only if for any Π' obtained from Π by a sequence of odd reflections, Y_α is locally nilpotent on $L(\lambda)$ for all $\alpha \in \Pi'$.*

Proof In one direction the statement is trivial. We need to prove the statement in the other direction. So assume that for any Π' obtained from Π by odd reflections, Y_α is locally nilpotent on $L(\lambda)$ for all $\alpha \in \Pi'$. Let β be a real root positive with respect to Π. Then $\mathfrak{g}_\beta$ acts locally nilpotently on $L(\lambda)$ for any λ, and we only have to show that $\mathfrak{g}_{-\beta}$ is locally nilpotent on $L(\lambda)$. By definition there exists a base Π'' such that β or $\frac{\beta}{2} \in \Pi''$. By Theorem 1, $\Pi'' = r_{\alpha_1}\cdots r_{\alpha_k}(\Pi)$ for some chain of reflections $r_{\alpha_1},\ldots,r_{\alpha_k}$. According to Exercise 2.6, we can assume without loss of generality that the reflections $r_{\alpha_1},\ldots,r_{\alpha_s}$ are even and the reflections $r_{\alpha_{s+1}},\ldots,r_{\alpha_k}$ are odd. Let

$$\Pi' = r_{\alpha_{s+1}}\cdots r_{\alpha_k}(\Pi)$$

and

$$\beta = r_{\alpha_1}\cdots r_{\alpha_s}(\alpha)$$

for some $\alpha \in \Pi'$. Since $\alpha_s \in \Pi'$, we have that $X_{\alpha_s}, Y_{\alpha_s}$ act locally nilpotently on $L(\lambda)$. Therefore

$$r_{\alpha_s} = \exp X_{\alpha_s} \exp(-Y_{\alpha_s}) \exp X_{\alpha_s}$$

is a well-defined linear operator on $L(\lambda)$. Similarly proceeding by induction, we obtain that $w = r_{\alpha_1} \cdots r_{\alpha_s}$ acts on $L(\lambda)$, and hence $\mathfrak{g}_{-\beta} = w^{-1}\mathfrak{g}_{-\alpha}w$ acts locally nilpotently on $L(\lambda)$. □

For an arbitrary base Σ, we denote by $L_\Sigma(\lambda)$ (resp., $M_\Sigma(\lambda)$) the simple $\mathfrak{g}$-module (resp., Verma module) with highest λ with respect to the triangular decomposition associated with Σ. From the above one can easily obtain that for any $w \in W$, if $\Sigma = w(\Pi)$ and λ is dominant integral, then

$$L_\Sigma\big(w(\lambda)\big) \simeq L_\Pi(\lambda).$$

Moreover, we have a similar identity for an odd reflection.

Lemma 3.5 *Let $\lambda \in \mathfrak{h}^*$ (not necessarily dominant integral), and $\Sigma = r_\alpha(\Pi)$ for some odd reflection r_α. Then*

$$L_\Sigma(\lambda') \simeq L_\Pi(\lambda),$$

where $\lambda' = \lambda - \alpha$ if $\lambda(h_\alpha) \neq 0$ and $\lambda' = \lambda$ if $\lambda(h_\alpha) = 0$.

Proof Let v be the highest vector of $L_\Pi(\lambda)$. If $\lambda(h_\alpha) = 0$, then $Y_\alpha v = 0$, and therefore v is annihilated by all simple roots of Σ. Hence v is a highest vector with respect to Σ. If $\lambda(h_\alpha) \neq 0$, then $Y_\alpha v$ is a highest vector with respect to Σ. □

Corollary 3.6 *A weight λ is dominant integral if and only if for any Σ obtained from Π by a sequence of odd reflections and any $\beta \in \Sigma$ such that $\beta(h_\beta) = 2$, we have $\lambda'(h_\beta) \in \mathbb{Z}_{\geq 0}$ if β is even and $\lambda'(h_\beta) \in 2\mathbb{Z}_{\geq 0}$ if β is odd, where $L_\Sigma(\lambda') \simeq L_\Pi(\lambda)$.*

The above corollary allows one to describe the set of all dominant integral weights (see, for example, [19, 26, 39]). Unfortunately if $\mathfrak{g}$ is infinite dimensional, the set of all integral dominant weights is rather small.

Theorem 5 (Kac–Wakimoto, Hoyt) *Let $\mathfrak{g}$ be an infinite-dimensional Kac–Moody Lie superalgebra (which is not a Lie algebra) of finite growth such that there exist an integrable $L(\lambda)$ of dimension greater than 1. Then $\mathfrak{g}$ is isomorphic to $\mathfrak{osp}(1|2n)^{(1)}$, $\mathfrak{osp}(2|2n)^{(1)}$, $\mathfrak{osp}(2|2n)^{(2)}$, $\mathfrak{sl}(1|n)^{(1)}$, $\mathfrak{sl}(1|n)^{(2)}$, or $S(2,1;b)$.*

A complete proof can be found in [15]. Here we give a sketch of the proof for $\mathfrak{g} = \mathfrak{s}^{(1)}$. Let $\mathfrak{s} \neq \mathfrak{osp}(1|2n), \mathfrak{osp}(2|2n)$, or $\mathfrak{sl}(1|n)$. Assume in addition that $\mathfrak{s} \neq D(2,1;a)$. Then $[\mathfrak{s}_0, \mathfrak{s}_0] = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ is a direct sum of two simple ideals. The subalgebra generated by all even real roots is isomorphic to $\mathfrak{s}_1^{(1)} \oplus \mathfrak{s}_2^{(1)}/\mathsf{k}K$, where

$K = aK_1 + bK_2$, K_1 and K_2 are the central elements of $\mathfrak{s}_1^{(1)}$ and $\mathfrak{s}_2^{(1)}$, respectively, and a, b are some positive rational numbers. Suppose that there exists an integrable $L(\lambda)$ with $\dim L(\lambda) > 1$. Then both $\lambda(K_1)$ and $\lambda(K_2)$ are positive (see [25]), but $\lambda(K) = a\lambda(K_1) + b\lambda(K_2) = 0$, a contradiction.

Since in the infinite-dimensional case there are very few integrable modules, one should try to generalize the notion of integrability. One possible generalization with very interesting applications is given in [26].

3.5 On the Weyl Character Formula

Let us introduce

$$D_0 = \prod_{\alpha \in \Delta_0^+} \left(1 - e^{-\alpha}\right)^{m(\alpha)},$$

$$D_1 = \prod_{\alpha \in \Delta_1^+} \left(1 + e^{-\alpha}\right)^{m(\alpha)},$$

where $m(\alpha) = \dim \mathfrak{g}_\alpha$. Recall that if $\mathfrak{g}$ is a symmetrizable Kac–Moody Lie algebra and $L(\lambda)$ is an integrable highest weight $\mathfrak{g}$-module, then

$$\operatorname{ch} L(\lambda) = \frac{e^\rho}{D_0} \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)}. \tag{6}$$

If $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, this formula is due to Hermann Weyl. For infinite-dimensional symmetrizable Kac–Moody Lie algebras it is proven by Kac [25]. This proof is quite remarkable in its simplicity and uses only the existence of the Casimir operator and the invariance of $\operatorname{ch} L(\lambda)$ under the action of the Weyl group. In the supercase, a similar formula was first obtained by Kac [21] in the case where the Cartan matrix of $\mathfrak{g}$ does not have zeros on the diagonal

$$\operatorname{ch} L(\lambda) = \frac{D_1 e^\rho}{D_0} \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)}. \tag{7}$$

The proof is practically the same as in the purely even case.

In general, however, formula (7) does not hold for all integrable highest weight modules. That leads us to the notion of a typical highest weight. We assume that $\mathfrak{g}$ is symmetrizable and call a real root α *isotropic* if $(\alpha, \alpha) = 0$. (Check that an isotropic real root is always odd). A weight λ is called *typical* if $(\lambda + \rho, \alpha) \neq 0$ for any isotropic real root α.

Theorem 6 *Let λ be a typical dominant integral weight. Then the character of $L(\lambda)$ is given by formula* (7).

Proof We will sketch a proof of the theorem in case where $(\alpha, \alpha) \geq 0$ for any real root α of $\mathfrak{g}$. This condition implies that $\mathfrak{g} = \mathfrak{osp}(1|2n)$, $\mathfrak{osp}(2|2n)$, $\mathfrak{sl}(1|n)$, or their (twisted) affinization. Note that, by Theorem 5, this implies the theorem for all infinite-dimensional symmetrizable Kac–Moody superalgebras. A general proof for finite-dimensional superalgebras will be given in the next section (it was first proven in [20]).

We start with the following simple observation.

Exercise 3.8

$$\operatorname{ch} M(\mu) = \frac{e^{\mu} D_1}{D_0}.$$

Exercise 3.9 *Let $U(\lambda)$ denote the right-hand side of (7). Check that $U(\lambda)$ is W-invariant and that*

$$U(\lambda) = \sum_{w \in W} \operatorname{sgn}(w) \operatorname{ch} M\big(w(\lambda + \rho) - \rho\big).$$

Introduce now the partial order $\leq$ on $\mathfrak{h}^*$ by putting $\lambda \leq \mu$ if $\mu - \lambda \in Q^+$.

Exercise 3.10 *One can write the character of any Verma module in a unique way as*

$$\operatorname{ch} M(\lambda) = \sum_{\mu \leq \lambda} b_\mu \operatorname{ch} L(\mu).$$

Moreover $b_\lambda = 1$, and if $b_\mu \neq 0$, then $(\lambda + 2\rho, \lambda) = (\mu + 2\rho, \mu)$.

Exercise 3.11 *If λ is typical dominant integral, then $\frac{2(\lambda+\rho,\beta)}{(\beta,\beta)} > 0$ for any non-isotropic real positive β.*

The above exercise immediately implies that the character of $L(\lambda)$ can be written in the form

$$\operatorname{ch} L(\lambda) = \sum_{\mu \leq \lambda} c_\mu \operatorname{ch} M(\mu) \tag{8}$$

with $c_\lambda = 1$. If $c_\mu \neq 0$, then $(\lambda + 2\rho, \lambda) = (\mu + 2\rho, \mu)$.

Now we recall that λ is integral dominant. Therefore $\operatorname{ch} L(\lambda)$ is invariant with respect to the W-action, and hence $\operatorname{ch} L(\lambda) - U(\lambda)$ is W-invariant. By the above exercises,

$$\operatorname{ch} L(\lambda) - U(\lambda) = \sum_{\nu \in F} a_\nu e^{\nu} = \sum_{\mu \in T} d_\mu \operatorname{ch} M(\mu).$$

By Exercise 3.8 and Exercise 3.11, $\nu \leq \lambda$ for any $\nu \in F$. By (8) $\mu < \lambda$ and $(\lambda + 2\rho, \lambda) = (\mu + 2\rho, \mu)$ for any $\mu \in T$.

Now let $\Sigma = r_{\alpha_1} \cdots r_{\alpha_k}(\Pi)$ for some odd reflections $r_{\alpha_1}, \ldots, r_{\alpha_k}$ and $\eta' = \eta - \alpha_1 - \cdots - \alpha_k$. Then $L_\Sigma(\lambda') \simeq L(\lambda)$ and $\operatorname{ch} M(\mu) = \operatorname{ch} M_\Sigma(\mu')$, and therefore

$$\lambda' - \mu' = \lambda - \mu = \sum_{\alpha \in \Sigma} m_\alpha \alpha \tag{9}$$

with some $m_\alpha \geq 0$.

Choose a maximal $\mu \in F$ with respect to $\leq$. Then $\mu \in T$ and, since F is W-invariant, $r_\beta(\mu) \leq \mu$. Therefore $\frac{2(\mu+\rho,\beta)}{(\beta,\beta)} \geq 0$ for any nonisotropic positive real root β. Since (9) holds for any Σ obtained from Π by odd reflections, it is not hard to see that

$$\lambda - \mu = \sum_{\beta \in \mathcal{B}} n_\beta \beta,$$

where $n_\beta \geq 0$, and $\mathcal{B}$ denotes the set of positive nonisotropic real roots.

The condition $(\lambda + 2\rho, \lambda) = (\mu + 2\rho, \mu)$ implies

$$\left(\lambda + \mu + 2\rho, \sum_{\beta \in \mathcal{B}} n_\beta \beta\right) = 0.$$

Furthermore, $(\beta, \beta) > 0$ by the initial assumptions on $\mathfrak{g}$, $(\lambda + \rho, \beta) > 0$, $(\rho, \beta) \geq 0$ by Exercise 3.11, and $(\mu, \beta) \geq 0$ by the maximality of μ. Therefore $n_\beta = 0$ for all $\beta \in \mathcal{B}$ and $\mu = \lambda$. On the other hand, $\mu < \lambda$, hence F is empty. The proof of the theorem is complete. □

The problem of finding $\operatorname{ch} L(\lambda)$ for any dominant integral λ is in fact rather complicated. In general, there is no nice formula for $\operatorname{ch} L(\lambda)$. There is however the following generic character formula which first appeared in [2]. Let $A(\lambda)$ be a maximal set of mutually orthogonal linearly independent positive real isotropic roots α such that $(\lambda + \rho, \alpha) = 0$. Set

$$S(\lambda) = \frac{D_1 e^\rho}{D_0} \sum_{w \in W} \operatorname{sgn}(w) w \left(\frac{e^{\lambda+\rho}}{\prod_{\alpha \in A(\lambda)} (1 + e^{-\alpha})} \right).$$

It was proven that $\operatorname{ch} L(\lambda) = S(\lambda)$ for $\mathfrak{g} = \mathfrak{sl}(1|n)$ in [2] and for $\mathfrak{g} = \mathfrak{osp}(2, 2n)$ in [17]. For the affine superalgebras $\mathfrak{sl}(1|n)^{(1)}$ and $\mathfrak{osp}(2|2n)^{(1)}$, a similar result is proven in [39]. In [18] this formula is proven for $\mathfrak{sl}(m|n)$ under the assumption that λ satisfies additional conditions. Finally, it is proven in [32] that $\operatorname{ch} L(\lambda) = S(\lambda)$ for generic λ.

To finish the discussion, let us mention that for the trivial module, the Weyl character formula is called the denominator identity. In the supercase the denominator identity was conjectured in [26] and proved in [8, 9].

4 Representation Theory of Finite-Dimensional Kac–Moody Superalgebras

In this section we assume that $\mathfrak{g}$ is a quasisimple finite-dimensional Kac–Moody superalgebra, i.e., $\mathfrak{g} = \mathfrak{sl}(m|n)$ with $m \neq n$, $\mathfrak{gl}(n|n)$, $\mathfrak{osp}(m|2n)$, $D(2,1,a)$, $G(1|2)$, or $F(1|3)$.

4.1 The Center of the Universal Enveloping Algebra and the Harish-Chandra Map

Let $Z(\mathfrak{g})$ denote the center of the universal enveloping algebra $U(\mathfrak{g})$. Fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

By the Poincaré–Birkhoff–Witt theorem,

$$U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+).$$

Let $\theta : U(\mathfrak{g}) \to U(\mathfrak{h})$ denote the projection with kernel $\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+$. Then the restriction of θ to $Z(\mathfrak{g})$ is a homomorphism of rings $Z(\mathfrak{g}) \to U(\mathfrak{h})$. It is called the Harish-Chandra homomorphism. Since $\mathfrak{h}$ is an abelian Lie algebra, $U(\mathfrak{h}) \simeq S(\mathfrak{h})$ can be considered as the algebra of polynomial functions on $\mathfrak{h}^*$. For any $\lambda \in \mathfrak{h}^*$, define the homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathsf{k}$ by

$$\chi_\lambda(z) = \theta(z)(\lambda).$$

Exercise 4.1 *Show that for any* λ, *the center* $Z(\mathfrak{g})$ *acts via* χ_λ *on the Verma module* $M(\lambda)$, *i.e.*,

$$zm = \chi_\lambda(z)m$$

for any $z \in Z(\mathfrak{g})$ *and* $m \in M(\lambda)$.

Exercise 4.2 *Show that* $\theta : Z(\mathfrak{g}) \to S(\mathfrak{h})$ *is injective.*

If $\mathfrak{g}$ is a semisimple Lie algebra, then a famous theorem of Harish-Chandra claims that the image of θ coincides with the algebra $S(\mathfrak{h})^W$ of all W-invariant polynomial functions on $\mathfrak{h}$ with respect to the shifted W-action: $w \cdot \lambda = w(\lambda + \rho) - \rho$. Since W is generated by reflections, $S(\mathfrak{h})^W \simeq Z(\mathfrak{g})$ is isomorphic to the algebra of polynomials in $\operatorname{rank}\mathfrak{g}$ variables.

In order to describe the image of θ in the supercase, define

$$Z(\lambda) = \{\mu \in \mathfrak{h}^* | \chi_\lambda = \chi_\mu\}.$$

Recall that $A(\lambda)$ is a maximal set of mutually orthogonal linearly independent isotropic roots α such that $(\lambda + \rho, \alpha) = 0$.

Theorem 7

$$Z(\lambda) = \bigcap_{w \in W} w \cdot \Bigl(\lambda + \bigoplus_{\alpha \in A(\lambda)} \mathsf{k}\alpha\Bigr).$$

We will not give a complete proof of this theorem here, but we will try to explain the main ideas.

Lemma 4.1 *Let $p \in S(\mathfrak{h}^*)$ lie in the image of θ. Then $p(\lambda) = p(w \cdot \lambda)$ for any $\lambda \in \mathfrak{h}^*$, $w \in W$, and if $(\lambda + \rho, \alpha) = 0$ for some isotropic root α, then $p(\lambda) = p(\lambda + c\alpha)$ for all $c \in \mathsf{k}$.*

Proof It suffices to show that $\chi_\lambda = \chi_{w\cdot\lambda}$ and that $\chi_{\lambda+c\alpha} = \chi_\lambda$ whenever $(\lambda + \rho, \alpha) = 0$ for an isotropic root α.

Let $w = r_\beta$ for some nonisotropic simple root β. Assume that $k = \frac{2(\lambda+\rho,\beta)}{(\beta,\beta)} \in \mathbb{N}$ and that k is odd if β is odd. A simple calculation shows that if v is the highest vector of $M(\lambda)$, then $Y_\beta^k v$ is annihilated by all simple roots. Therefore there exists a nontrivial homomorphism $M(r_\beta \cdot \lambda) \to M(\lambda)$. Hence $\chi_\lambda = \chi_{r_\beta\cdot\lambda}$ for all λ satisfying our assumptions. But the set of all λ satisfying these assumptions is Zariski dense, therefore $\chi_\lambda = \chi_{r_\beta\cdot\lambda}$ for all λ.

If β is a nonisotropic root which belongs to some base Σ obtained from the original base Π by odd reflections, and if λ is typical, then repeating the above arguments and using $M_\Sigma(\lambda') \simeq M_\Pi(\lambda)$ with λ' defined as in the proof of Theorem 6, one obtains $\chi_\lambda = \chi_{r_\beta\cdot\lambda}$ for all λ. Finally the Weyl group W is generated by all such r_β, hence we have $\chi_\lambda = \chi_{w\cdot\lambda}$ for all $w \in W$ and all $\lambda \in \mathfrak{h}^*$.

To check the second condition for an isotropic α, we first observe that W acts transitively on the set of isotropic roots. Therefore it suffices to prove in the case where α is a simple isotropic root. In this case if $(\lambda + \rho, \alpha) = 0$, there is a nonzero homomorphism $M(\lambda - \alpha) \to M(\lambda)$, hence $\chi_\lambda = \chi_{\lambda-\alpha}$. But then $\chi_\lambda = \chi_{\lambda-k\alpha}$ for any negative integer k. Again by a Zariski density argument, we have $\chi_\lambda = \chi_{\lambda-c\alpha}$ for any $c \in \mathsf{k}$. □

Next we have to show that every $p \in S(\mathfrak{h})$ satisfying the conditions of Lemma 4.1 belongs to the image of θ. This is the most difficult part of the proof, and we omit it here. It is done in [40] and in [10] by two different methods, and we refer the reader to these papers. The method of [10] has been originally announced in [23].

To finish the proof of Theorem 7, one should check that every $p \in S(\mathfrak{h})$ satisfies the conditions of Lemma 4.1 if and only if p is constant on $Z(\lambda)$ for all λ. This is an exercise which we leave to the reader.

The method used in [40] is similar to the one in the proof of the classical Harish-Chandra theorem. One can identify $U(\mathfrak{g})$ with $S(\mathfrak{g}^*)$ by the invariant symmetric form on $\mathfrak{g}$. Under this identification, $Z(\mathfrak{g})$ corresponds to the subspace $S(\mathfrak{g}^*)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants under the adjoint action, and the Harish-Chandra homomorphism corresponds to the restriction map $S(\mathfrak{g}^*) \to S(\mathfrak{h}^*)$. The ring $S(\mathfrak{g}^*)^{\mathfrak{g}}$ is generated by the traces of all finite-dimensional modules of $\mathfrak{g}$.

Example Let $\mathfrak{g} = \mathfrak{sl}(m|n)$. It was shown in [40] that $S(\mathfrak{g}^*)^{\mathfrak{g}}$ is generated by $p_n(x) = \mathrm{str}(X^k)$ for all $k \geq 2$. In contrast with the case $\mathfrak{g} = \mathfrak{sl}(n)$, the ring $S(\mathfrak{g}^*)^{\mathfrak{g}}$ is not Noetherian.

An important corollary of Theorem 7 is the following.

Corollary 4.2 *If* λ *is typical, then* $Z(\lambda) = W \cdot \lambda$.

Corollary 4.3 *If* λ *is typical, then* $L(\lambda)$ *is projective and injective in the category of finite-dimensional weight* $\mathfrak{g}$*-modules.*

Note that Corollary 4.2 easily implies Theorem 6 for all finite-dimensional $\mathfrak{g}$. Indeed, as in Exercise 3.10, one can show that

$$\mathrm{ch}\, L(\lambda) = \sum_{\mu \in Z(\lambda)} c_\mu \,\mathrm{ch}\, M(\mu).$$

By Corollary 4.2 we obtain

$$\mathrm{ch}\, L(\lambda) = \sum_{w \in W} c_w \,\mathrm{ch}\, M(w \cdot \lambda) = \frac{D_1 e^\rho}{D_0} \sum_{w \in W} c_w e^{w(\lambda+\rho)}.$$

The W-invariance of $\mathrm{ch}\, L(\lambda)$ implies $c_w = \mathrm{sgn}(w) c_e$. In addition $c_e = c_\lambda = 1$, hence Theorem 6.

The *defect of* $\mathfrak{g}$ (notation $\mathrm{def}\,\mathfrak{g}$) is a maximal number of linearly independent mutually orthogonal isotropic roots.

Exercise 4.3 $\mathrm{def}\,\mathfrak{sl}(m|n) = \min\{m, n\}$.

For any weight λ, the *degree of atypicality* (notation $\mathrm{at}(\lambda)$) is the cardinality of $A(\lambda)$. Obviously $\mathrm{at}(\lambda) \leq \mathrm{def}\,\mathfrak{g}$.

Exercise 4.4 *If* $\mu \in Z(\lambda)$, *then* $\mathrm{at}(\lambda) = \mathrm{at}(\mu)$. *Hence for any* $\chi : Z(\mathfrak{g}) \to \mathsf{k}$, $\mathrm{at}(\chi)$ *is well defined.*

For any vector superspace V, set $\mathrm{sdim}(V) = \dim V_0 - \dim V_1$.

Conjecture 4.4 [26] *Let* λ *be an integral dominant weight. Then* $\mathrm{sdim}(L(\lambda)) \neq 0$ *if and only if* $\mathrm{at}(\lambda) = \mathrm{def}\,\mathfrak{g}$.

4.2 Associated Variety

Here we introduce the notion of associated variety and review some results of [7].

Let G_0 denote a reductive algebraic group with Lie algebra $\mathfrak{g}_0$. Let

$$X = \left\{x \in \mathfrak{g}_1 \mid [x, x] = 0\right\}.$$

Obviously, X is a G_0-invariant Zariski closed set in $\mathfrak{g}_1$. We refer to X as the self-commuting cone in $\mathfrak{g}_1$.

Example 4.5 In the case where $\mathfrak{g} = \mathfrak{sl}(m|n)$, the self-commuting cone X consists of matrices

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

such that $AB = BA = 0$.

Theorem 8 [7] *Every G_0-orbit on X contains an element $x = X_{\alpha_1} + \cdots + X_{\alpha_k}$ for some set of linearly independent mutually orthogonal isotropic roots $\{\alpha_1, \ldots, \alpha_k\}$. Let S be the family of all sets of linearly independent mutually orthogonal isotropic roots. Then we have a bijection between the set of W-orbits in S and the set of G_0-orbits in X. In particular, the set of G_0-orbits on X is finite.*

Exercise 4.6 *Prove Theorem 8 for $\mathfrak{g} = \mathfrak{sl}(m|n)$.*

As follows from Theorem 8, every x lies on the G_0-orbit of $X_{\alpha_1} + \cdots + X_{\alpha_k}$ for some $\{\alpha_1, \ldots, \alpha_k\} \in S$. The number k is called the *rank* of x. By X_k we denote the set of all $x \in X$ of rank k. Thus, we get a stratification

$$X = \bigcup_{k \leq \operatorname{def} \mathfrak{g}} X_k, \qquad \bar{X}_k = \bigcup_{j \leq k} X_j.$$

Let $x \in X \subset \mathfrak{g}$, and M be a $\mathfrak{g}$-module. Since $[x, x] = 0$, x^2 annihilates M. Set

$$M_x = \ker x / \operatorname{Im} x.$$

The *associated variety* X_M of M is defined by setting

$$X_M = \{x \in X \mid M_x \neq 0\}.$$

It is clear that if M is finite dimensional, then X_M is a G_0-invariant Zariski closed subset of X.

Exercise 4.7

(i) $X_{M \oplus N} = X_M \cup X_N$;
(ii) $X_{M \otimes N} = X_M \cap X_N$;
(iii) $\operatorname{sdim} M_x = \operatorname{sdim} M$.

Theorem 9 [7] *Let M be a finite-dimensional $\mathfrak{g}$-module with central character χ and* $\operatorname{at}(\chi) = k$. *Then $X_M \subset \bar{X}_k$.*

Proof Let $\mathfrak{z}_{\mathfrak{g}}(x)$ denote the centralizer of x in $\mathfrak{g}$. It is clear that $[x, \mathfrak{g}]$ is an ideal in $\mathfrak{z}_{\mathfrak{g}}(x)$. Let $\mathfrak{g}_x = \mathfrak{z}_{\mathfrak{g}}(x)/[x, \mathfrak{g}]$. It is not hard to show that $\mathfrak{g}_x$ is again a quasisimple Kac–Moody superalgebra of the same type as $\mathfrak{g}$. We may assume that $x = X_{\alpha_1} + \cdots + X_{\alpha_k}$ as in Theorem 8; then

$$\mathfrak{h}_x = (\ker \alpha_1 \cap \cdots \cap \ker \alpha_k)/(\mathsf{k}h_{\alpha_1} \oplus \cdots \oplus \mathsf{k}h_{\alpha_k}) \tag{10}$$

is the Cartan subalgebra of $\mathfrak{g}_x$, and

$$\Delta_x = \{\alpha \in \Delta \,|\, (\alpha, \alpha_i) = 0, \alpha \neq \pm\alpha_i, i = 1, \ldots, k\}$$

is the set of roots of $\mathfrak{g}_x$.

Exercise 4.8 *If* $\mathfrak{g} = \mathfrak{sl}(m|n)$ *and* $\mathrm{rk}(x) = k$, *then* $\mathfrak{g}_x \simeq \mathfrak{sl}(m-k|n-k)$.

It is clear that $\ker x$ is $\mathfrak{z}_{\mathfrak{g}}(x)$-invariant and that $[x, \mathfrak{g}] \ker_x \subset \mathrm{Im}\, x$. Thus, M_x has a natural structure of $\mathfrak{g}_x$-module.

Let $U(\mathfrak{g})^x$ be the subalgebra of ad_x-invariants. Then $[x, U(\mathfrak{g})]$ is an ideal in $U(\mathfrak{g})^x$, and

$$U(\mathfrak{g}_x) = U(\mathfrak{g})^x / [x, U(\mathfrak{g})].$$

Let ϕ denote the projection $U(\mathfrak{g})^x \to U(\mathfrak{g}_x)$. It is not difficult to see that $\phi(Z(\mathfrak{g})) \subset Z(\mathfrak{g}_x)$ and $\phi : Z(\mathfrak{g}) \to Z(\mathfrak{g}_x)$ is a homomorphism of rings. By $\check{\phi}$ we denote the dual map $\check{Z}(\mathfrak{g}_x) \to \check{Z}(\mathfrak{g})$, where $\check{A} = \mathrm{Hom}(A, \mathsf{k})$. It follows immediately from the construction that, if M admits central character χ, then M_x can admit only central characters in $\check{\phi}^{-1}(\chi)$.

Now we are going to describe $\check{\phi}$. For this, we again assume that $x = X_{\alpha_1} + \cdots + X_{\alpha_k}$. It is not difficult to check case by case that one can always find a base containing $\alpha_1, \ldots, \alpha_k$. We use the Harish-Chandra homomorphisms $\theta : Z(\mathfrak{g}) \to S(\mathfrak{h})$ and $\theta_x : Z(\mathfrak{g}_x) \to S(\mathfrak{h}_x)$ associated with the triangular decomposition defined by this base. Note that (10) implies

$$\mathfrak{h}_x^* = (\mathsf{k}\alpha_1 \oplus \cdots \oplus \mathsf{k}\alpha_k)^{\perp} / (\mathsf{k}\alpha_1 \oplus \cdots \oplus \mathsf{k}\alpha_k).$$

Let

$$p : (\mathsf{k}\alpha_1 \oplus \cdots \oplus \mathsf{k}\alpha_k)^{\perp} \to \mathfrak{h}_x^*$$

denote the natural projection. By Theorem 7, if $\nu, \nu' \in p^{-1}(\mu)$, then $\chi_\nu = \chi_{\nu'}$.

We claim that $\check{\phi}(\chi_\mu) = \chi_\nu$ for some $\nu \in p^{-1}(\mu)$. Indeed, let M be the quotient of the Verma module $M(\nu)$ by the submodule generated by vectors $Y_{\alpha_1} v, \ldots, Y_{\alpha_k} v$ (v as before stands for the highest vector). Then $v \in M_x$, and therefore M_x contains the Verma module over $\mathfrak{g}_x$ with highest weight μ. Hence $\chi_\mu \in \check{\phi}^{-1}(\chi_\nu)$.

From the above description we obtain that $\mathrm{at}(\check{\phi}^{-1}(\chi)) = \mathrm{at}(\chi) - k$ for any $\chi \in \check{Z}(\mathfrak{g})$. Therefore, if $\mathrm{rk} x = k > \mathrm{at}(\chi)$, we have $M_x = 0$. The proof of the theorem is complete. □

Note that Theorem 9 and Exercise 4.7(iii) imply Conjecture 4.4 in one direction: if $\mathrm{at}(\lambda) < \mathrm{def}(\mathfrak{g})$, then $\mathrm{sdim}\, L(\lambda) = 0$.

For other applications of X_M, see [7].

4.3 Geometric Induction (Cohomological Induction)

In this subsection we discuss some applications of the Zuckerman functor (see [43]) to the representation theory of finite-dimensional superalgebras.

For any $\mathfrak{g}$-module M, let $\Gamma(M)$ denote the subspace of all $\mathfrak{g}_0$-finite vectors. It is easy to see that $\Gamma(M)$ is a $\mathfrak{g}$-module and that Γ is a left exact functor in the category of $\mathfrak{g}$-modules. Let $\Gamma^i = R^i(\Gamma)$ denote the right derived functor of Γ (details of this definition can be found in [43]).

Pick $h \in \mathfrak{h}$ so that $\alpha(h)$ is rational for all roots α. Set

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha(h)=0} \mathfrak{g}_\alpha, \qquad \mathfrak{m}^+ = \bigoplus_{\alpha(h)>0} \mathfrak{g}_\alpha, \qquad \mathfrak{m}^- = \bigoplus_{\alpha(h)<0} \mathfrak{g}_\alpha.$$

Then $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}^+$ is a *parabolic* subalgebra according to the definition of [16], and $\mathfrak{l}$ is the *reductive part* (or Levi subalgebra) of $\mathfrak{p}$. If $\mathfrak{l} = \mathfrak{h}$, a parabolic subalgebra is called a *Borel subalgebra*.

Note that $\mathfrak{m}^+$ acts trivially on any irreducible finite-dimensional $\mathfrak{p}$-module. For any $\lambda \in \mathfrak{h}^*$, we denote by $L_\mathfrak{p}(\lambda)$ the irreducible $\mathfrak{p}$-module with highest weight λ. It is clear that $L_\mathfrak{p}(\lambda)$ is finite dimensional if λ is dominant integral with respect to $\mathfrak{l}$. In what follows we always assume that $L_\mathfrak{p}(\lambda)$ is finite dimensional.

For a finite-dimensional $\mathfrak{p}$-module V, define

$$H^i(G/P, V) = \Gamma^i\big(\mathrm{Hom}_{U(\mathfrak{p})}\big(U(\mathfrak{g}), V\big)\big).$$

The notation is related to the following geometric interpretation of Γ^i in this case. One can show that there exist a closed algebraic subsupergroup $P \subset G$ with Lie superalgebra $\mathfrak{p}$ and the homogeneous smooth algebraic supervariety G/P. If V is a P-module, then one can define the induced vector bundle $\mathcal{L}(V) = G \times_P V$. It is possible to show that the ith cohomology group of G/P with coefficients in the sheaf of sections of $\mathcal{L}(V)$ coincides with $H^i(G/P, V)$. In geometric terms, Γ^i was treated in [33] and later in [13, 37, 38]. The algebraic properties of Γ^i were studied in [36].

If $\mathfrak{g}$ is a semisimple Lie algebra, then $H^i(G/P, L_\mathfrak{p}(\lambda))$ are well known from the Borel–Weil–Bott theorem.

Theorem 10 (Borel–Weil–Bott) *Assume that $L_\mathfrak{p}(\lambda)$ is a finite-dimensional P-module. If $\lambda + \rho$ is not regular, then $H^i(G/P, L_\mathfrak{p}(\lambda)^*) = 0$ for all i. If $\lambda + \rho$ is regular then there exists a unique $w \in W$, such that $w(\lambda + \rho) - \rho$ is dominant. Let*

l denote the length of w. Then

$$H^i\big(G/P, L_{\mathfrak{p}}(\lambda)^*\big) = \begin{cases} 0, & \text{if } i \neq l, \\ L(\lambda)^*, & \text{if } i = l. \end{cases}$$

In general, the Borel–Weil–Bott theorem does not hold in the supercase. It is still true when λ is typical.

Theorem 11 [33] *If* $\mathrm{at}(\lambda) = 0$, *then the Borel–Weil–Bott theorem holds.*

We do not give a proof since it is rather technical and involves supergeometry. On the other hand, it is easy to see that if λ is a typical dominant integral weight, then $H^0(G/P, L_{\mathfrak{p}}(\lambda)^*) = L(\lambda)^*$. Indeed, it follows from the definition and from a standard duality argument showing that $H^0(G/P, L_{\mathfrak{p}}(\lambda)^*)^*$ is a finite-dimensional quotient of the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}(\lambda)$. In particular, it affords the central character χ_λ, and the multiplicity of the weight λ is 1. Hence $H^0(G/P, L_{\mathfrak{p}}(\lambda)^*)^* \simeq L(\lambda)$ by Corollary 4.3.

The following exercise illustrates what happens if λ is not typical.

Exercise 4.9 *Let* $G = \mathfrak{sl}(m|n)$, *and let* $\mathfrak{p}$ *consist of block matrices of the following type*:

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Then $H^i(G/P, L_{\mathfrak{p}}(\lambda)^*) = 0$ *for* $i > 0$, *and*

$$H^0\big(G/P, L_{\mathfrak{p}}(\lambda)^*\big) = \mathrm{Hom}_{U(\mathfrak{p})}\big(U(\mathfrak{g}), L^*_{\mathfrak{p}}(\lambda)\big).$$

This module is simple iff λ is typical.

Although in most cases we do not understand the structure of $H^i(G/P, V)$, we still have some useful information about its character.

Theorem 12 [33] *If V is a finite-dimensional* $\mathfrak{p}$*-module, then*

$$\sum_i (-1)^i \,\mathrm{ch}\big(H^i(G/P, V^*)\big)^* = \frac{e^\rho D_1}{D_0} \sum_{w \in W} \mathrm{sgn}(w) w \left(\frac{\mathrm{ch}\, V e^\rho}{\prod_{\alpha \in \Delta_1^+(\mathfrak{l})} (1 + e^{-\alpha})} \right).$$

Proof We sketch the proof in a sequences of exercises.

Exercise 4.10 *If* $\mathfrak{g}$ *is finite dimensional*, *then* $\rho = \rho_0 - \rho_1$, *where*

$$\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha, \qquad \rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha.$$

Exercise 4.11 *Show that if F is a finite-dimensional simple $\mathfrak{p}_0$-module, then the Borel–Weil–Bott theorem, together with the Weyl character formula, implies*

$$\sum_i (-1)^i \operatorname{ch}\bigl(H^i(G_0/P_0, F^*)\bigr)^* = \frac{e^{\rho_0}}{D_0} \sum_{w\in W} \operatorname{sgn}(w) w\bigl(\operatorname{ch} F e^{\rho_0}\bigr). \tag{11}$$

Exercise 4.12 *Let F be an arbitrary finite-dimensional $\mathfrak{p}_0$-module. Then the restriction of F to $\mathfrak{l}_0$ is isomorphic to a direct sum of simple modules $F = F_1 \oplus \cdots \oplus F_s$. Show that*

$$\sum_i (-1)^i \operatorname{ch}\bigl(H^i(G/P, F^*)\bigr) = \sum_{j=1}^{s} \sum_i (-1)^i \operatorname{ch}\bigl(H^i(G/P, F_j^*)\bigr)$$

and therefore that (11) holds for arbitrary finite-dimensional $\mathfrak{p}_0$-module F.

To finish the proof, let $M = S(\mathfrak{g}_1/\mathfrak{p}_1) \otimes U(\mathfrak{g}_0)$ with the natural structure of a left $\mathfrak{p}_0$-module and a right $\mathfrak{g}_0$-module. Then $U(\mathfrak{g}) \simeq U(\mathfrak{p}) \otimes_{U(\mathfrak{p}_0)} M$, and by Frobenius reciprocity we have an isomorphism of $\mathfrak{g}_0$-modules

$$\operatorname{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), V^*) \simeq \operatorname{Hom}_{U(\mathfrak{p}_0)}(M, V^*) \simeq \operatorname{Hom}_{U(\mathfrak{p}_0)}\bigl(U(\mathfrak{g}_0), \bigl(S(\mathfrak{g}_1/\mathfrak{p}_1) \otimes V\bigr)^*\bigr).$$

Let $F = S(\mathfrak{g}_1/\mathfrak{p}_1) \otimes V$. Then

$$\operatorname{ch} F = \operatorname{ch} V \frac{D_1}{\prod_{\alpha\in\Delta^+(\mathfrak{l})}(1+e^{-\alpha})},$$

and (11) implies

$$\sum_i (-1)^i \operatorname{ch}\bigl(H^i(G/P, V^*)\bigr)^* = \frac{e^{\rho_0}}{D_0} \sum_{w\in W} \operatorname{sgn}(w) w\left(\frac{\operatorname{ch} V e^{\rho_0} D_1}{\prod_{\alpha\in\Delta^+(\mathfrak{l})}(1+e^{-\alpha})}\right).$$

Since $e^{\rho_1} D_1$ is W-invariant and $\rho = \rho_0 - \rho_1$, the theorem now follows from a simple substitution. □

Finally, we cite without proof the following result (see [13, 37]).

Theorem 13 *Let $\mathfrak{g} = \mathfrak{sl}(m|n)$ (resp., $\mathfrak{osp}(m|2n)$), and $\mathfrak{p}$ be the parabolic subalgebra with $\mathfrak{l} \simeq \mathfrak{sl}(m-1|n) \oplus \mathsf{k}$ (resp., $\mathfrak{osp}(m-2|2n) \oplus \mathsf{k}$ or $\mathfrak{osp}(m|2n-2) \oplus \mathsf{k}$. Let λ be an integral dominant weight. Then the cardinality of $A(\lambda) \cap P(\mathfrak{m}^+)$ is at most one.*

(i) *If $A(\lambda) \cap P(\mathfrak{m}^+)$ is empty, then $H^i(G/P, L_{\mathfrak{p}}(\lambda)^*) = 0$ for $i > 0$ and $H^0(G/P, L_{\mathfrak{p}}(\lambda)^*) = L(\lambda)^*$.*
(ii) *If $|A(\lambda) \cap P(\mathfrak{m}^+)| = 1$, then $H^i(G/P, L_{\mathfrak{p}}(\lambda)^*)$ is semisimple and multiplicity-free for $i > 0$, $H^0(G/P, L_{\mathfrak{p}}(\lambda)^*)$ has a unique irreducible submodule isomorphic to $L(\lambda)^*$, and the quotient by this submodule is a semisimple multiplicity free $\mathfrak{g}$-module.*

The proof of this theorem is based on a rather complicated combinatorial algorithm that gives a complete description of $H^i(G/P, L_{\mathfrak{p}}(\lambda)^*)$. Combining this algorithm with Theorem 12, one can obtain a combinatorial algorithm calculating $\mathrm{ch}\, L(\lambda)$ (see [37] and [13]).

Another very interesting approach to the problem of calculating $\mathrm{ch}\, L(\lambda)$ can be found in [3] and [5] for $\mathfrak{sl}(m|n)$ and in the recent paper [6] for $\mathfrak{osp}(m|2n)$.

4.4 Open Problems

The problem of finding the character of a simple finite-dimensional $\mathfrak{g}$-module is solved for simple basic classical superalgebras $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(m|2n)$ as was explained in the end of the previous section. This problem is solved also for $\mathfrak{q}(n)$ in [35] and [4], and for Cartan type superalgebras in [1] and [41]. The cases of exceptional superalgebras and $\mathfrak{p}(n)$ are still open. Since all exceptional superalgebras have defect 1, one should expect that the Bernstein–Leites formula $\mathrm{ch}\, L(\lambda) = S(\lambda)$ holds in this case. The case of $\mathfrak{p}(n)$, however, requires new ideas. At this time only a generic character formula is known (see [34]).

Using methods of the two previous sections, the author managed to prove Conjecture 4.4 for $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(m|2n)$. It is interesting to formulate and prove this conjecture for all finite-dimensional simple Lie superalgebras.

An analogue of the Borel–Weil–Bott theorem is unknown for most atypical weights and most parabolic subalgebras.

Finally, it is interesting to obtain character formulas for irreducible integrable modules of the nonsymmetrizable Kac–Moody superalgebra $S(2, 1; b)$.

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Categories of Harish-Chandra Modules

Wolfgang Soergel

Abstract We discuss the conjectural relation between the structure of a category of representations and the geometry of its space of Langlands parameters, emphasizing examples.

Keywords Koszul duality · Langlands philosophy · Kazhdan–Lusztig · Real reductive group

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1 Introduction to Soergel's Conjecture

Our story begins with a pair consisting of a complex reductive algebraic group G and an antiholomorphic involution γ, also known as a *real form* of G, on it. We will assume that γ is quasi-split; in other words, that it fixes a Borel subgroup of G. We will use $\mathfrak{g}$ to denote the Lie algebra of G. From $\mathfrak{g}$ we construct its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ in the usual way. For the center Z of $\mathcal{U}(\mathfrak{g})$, we fix a character $\chi \in \mathit{Char}\, Z$. Then, to the data (G, γ, χ), we will generate two pictures of representation-theoretic data which will be related to one another by Koszul duality.

The first picture is of irreducible representations of the real points of γ,

$$G(\mathbb{R}, \gamma) := G^\gamma = \{g \in G | g^\gamma = g\}.$$

We use $\mathit{Irr}(G(\mathbb{R}, \gamma))_\chi$ to denote the set of isomorphism classes of irreducible Harish-Chandra modules for $G(\mathbb{R}, \gamma)$ with Z acting by the character χ. In fact, we will actually consider an entire collection of strong real forms inner to γ:

$$\bigsqcup_{\delta \in \mathrm{H}^1(\Gamma; G)} \mathit{Irr}\big(G(\mathbb{R}, \delta)\big)_\chi, \tag{1}$$

where the nontrivial element of $\Gamma = \mathit{Gal}(\mathbb{C}/\mathbb{R})$ acts on G by γ, and $\mathrm{H}^1(\Gamma; G)$ is the first Galois cohomology group of Γ with coefficients in G. The usual homomorphism $G \to \mathit{Aut}(G)$ sending $g \in G$ to the inner automorphism determined by g

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allows us to map $\mathrm{H}^1(\Gamma; G) \to \mathrm{H}^1(\Gamma; Aut(G))$ and thereby consider $\delta \in \mathrm{H}^1(\Gamma; G)$ to be the real form defined by the conjugation $g^\delta = \delta g^\gamma \delta^{-1}$ for any $g \in G$. The group of real points for δ is then the fixed points of the action

$$G(\mathbb{R}, \delta) = \{g \in G | g^\delta = g\}.$$

The second picture is geometric in nature and occurs on the dual side. To the data (G, γ, χ), the authors of [ABV92] associate a complex algebraic variety $X = X_{\gamma,\chi}$ on which $G^\vee$, the complex dual group of G, acts algebraically with a finite number of orbits. Recall that the quasi-split real form γ on G from above gives rise to a holomorphic involution γ on the dual group $G^\vee$. If χ is integral, for example, then

$$X_{\gamma,\chi} = G^\vee \times_{P^\vee(\chi)} Z^1(\Gamma; G^\vee),$$

where $P^\vee(\chi)$ is a parabolic in $G^\vee$ which depends on χ, and

$$Z^1(\Gamma; G^\vee) = \{g \in G^\vee | g(g^\gamma) = 1\}.$$

For this data, we define a collection of parameters

$$\begin{aligned} \mathrm{Par}(X) = \{(Y, \tau) | Y \subset X \text{ a } G^\vee\text{-orbit, } \tau \text{ an irreducible equivariant} \\ \mathbb{C}\text{-local system on } Y\}. \end{aligned} \tag{2}$$

For a fixed $G^\vee$ orbit Y in X, the irreducible equivariant local systems τ on Y are themselves parameterized by irreducible representations of the component group $G^\vee_y / G^{\vee\circ}_y$ of the isotropy group $G^\vee_y$ in $G^\vee$ of an arbitrary point $y \in Y$.

Following [ABV92], these seemingly unrelated parameter spaces are in fact in bijection

$$\bigsqcup_\delta Irr\big(G(\mathbb{R}, \delta)\big)_\chi \xrightarrow{\sim} \mathrm{Par}(X). \tag{3}$$

Additional input can be introduced to establish a more precise relationship between Harish-Chandra modules and the geometric parameters appearing on the dual side. Let $\pi \in \mathrm{Par}(X)$, and let $\mathscr{L}_\pi \in Irr(G(\mathbb{R}, \delta))_\chi$ be the irreducible Harish-Chandra module associated to it by the bijection (3). Any irreducible Harish-Chandra module such as $\mathscr{L}_\pi$ appears as the unique simple quotient

$$\mathscr{M}_\pi \twoheadrightarrow \mathscr{L}_\pi \tag{4}$$

of a standard representation $\mathscr{M}_\pi$ also determined by the parameter π. The Jordan–Hölder matrix JH for (G, γ, χ) has entries given by the multiplicities of $\mathscr{L}_{\pi'}$ in the composition series of the standard modules $\mathscr{M}_\pi$ for $(\pi, \pi') \in \mathrm{Par}(X) \times \mathrm{Par}(X)$. That is to say,

$$JH_{(\pi,\pi')} = [\mathscr{M}_\pi : \mathscr{L}_{\pi'}]. \tag{5}$$

In the picture of [ABV92], we can define a second matrix associated directly to the geometric parameters, which we will call the intersection cohomology matrix and denote by IC. If $j : Y \to X$ is the inclusion of the $G^\vee$ orbit Y and $\pi = (Y, \tau)$, define $\mathscr{M}^\pi := j_! \tau$, the exceptional pushforward of the local system τ. This module has a unique simple submodule $\mathscr{L}^\pi$ which is the intersection cohomology complex $\mathscr{L}^\pi = j_{!*}\tau$. We may also use IC_π for this irreducible module. Then, the intersection cohomology matrix is defined to have (π, π') entry

$$IC_{(\pi,\pi')} = \left[\mathscr{M}^\pi : \mathscr{L}^{\pi'}\right]. \tag{6}$$

Up to the signs of the entries, the JH and IC matrices are related by taking the inverse transpose. This is essentially a reformulation of the results of one of Vogan's earlier papers [Vog82]. Observe that the disjoint union of the $Irr(G(\mathbb{R}, \delta))_\chi$ forces JH to be block diagonal. The identification of IC with the inverse transpose of JH then forces IC to also be block diagonal—a fact which was not so transparent initially. Syu Kato has a geometric explanation for the block decomposition of IC.

With the above identification of the combinatorial data arising in each picture, the next natural question is whether there is some categorical equivalence informing the bijection of parameters and the relationship between the JH and IC matrices. We give an incomplete formulation of the conjectured categorical equivalence here; more details can be found in [Soe01]. Let $\mathcal{M}(G(\mathbb{R}, \delta))_\chi$ denote the category of finite length Harish-Chandra modules for the real form $G(\mathbb{R}, \delta)$ which are annihilated by some power of χ. For the dual group, consider the sum of all the IC-complexes $\bigoplus_{\pi \in \mathrm{Par}(X)} IC_\pi$. Let $\mathrm{Ext}^\bullet_{G^\vee}(\bigoplus_\pi IC_\pi)$ be the graded $\mathbb{C}$-algebra of $G^\vee$-equivariant self-extensions of $\bigoplus_\pi IC_\pi$. We use $\mathrm{Ext}^\bullet_{G^\vee}(\bigoplus_\pi IC_\pi)$ instead of $\mathrm{Ext}^\bullet_{G^\vee}(\bigoplus_\pi IC_\pi, \bigoplus_\pi IC_\pi)$ for the sake of brevity. The modules over $\mathrm{Ext}^\bullet_{G^\vee}(\bigoplus_\pi IC_\pi)$ which are finite dimensional and annihilated by high degrees form the objects of a category $\mathrm{Ext}^\bullet_{G^\vee}(\bigoplus_\pi IC_\pi)$-$\mathcal{N}il$.

Conjecture

$$\bigoplus_{\delta \in \mathrm{H}^1(\Gamma; G)} \mathcal{M}\big(G(\mathbb{R}, \delta)\big)_\chi \simeq \mathrm{Ext}^\bullet_{G^\vee}\left(\bigoplus_\pi IC_\pi\right)\text{-}\mathcal{N}il \tag{7}$$

The conjecture is known among others to be true for tori, complex algebraic groups, for generic central character, and for $SL(2, \mathbb{R})$.

Example 1 Consider the case $G = \mathbb{C}^\times$, $\gamma : z \mapsto \bar{z}$. Then, $\mathfrak{g} = \mathbb{C}$, and γ acts as the identity on $G^\vee = \mathbb{C}^\times$. Therefore, $Z^1(\Gamma; G^\vee) = \{\pm 1\}$, and consequently $X_{\gamma,\chi}$ is two points. Thus, the equivariant self-extension ring on the right-hand side of the conjecture is

$$\mathrm{H}^\bullet_{\mathbb{C}^\times}(X_{\gamma,\chi}) = \mathbb{C}[t] \times \mathbb{C}[t], \tag{8}$$

the $\mathbb{C}^\times$-equivariant cohomology of two points. What are the real forms δ? The first Galois cohomology $\mathrm{H}^1(\Gamma; G)$ happens to be trivial in this example. The trivial class in $\mathrm{H}^1(\Gamma; G)$ always corresponds to the action of γ itself, so the left-hand side is

$$\mathcal{M}\big(G(\mathbb{R}, \gamma)\big)_\chi = \mathcal{M}\big(\mathbb{R}^\times\big)_\chi.$$

Moreover, the category $\mathcal{M}(\mathbb{R}^\times) \simeq (\mathbb{C}[x] \times \mathbb{C}[x])\text{-}mod^{fd}$, the category of finite-dimensional modules over $\mathbb{C}[x] \times \mathbb{C}[x]$. The polynomial algebra $\mathbb{C}[x]$ is the universal enveloping algebra of the complexification of the Lie algebra $Lie(\mathbb{R}^\times) \simeq \mathbb{R}$. Two copies of $\mathbb{C}[x]$ appear since $\mathbb{R}^\times$ is disconnected. To explain the parameter t on the other side, note that for T a complex torus, the T-equivariant cohomology of a point $\mathrm{H}^\bullet_T(pt) = \mathcal{O}_{Lie(T)}$, the ring of regular functions on the Lie algebra of T, with its $\mathbb{Z}$-grading doubled. So t is a linear form on the Lie algebra of the dual torus, whereas x is an element of the Lie algebra of the original torus, which means that indeed there is a canonical identification of our two polynomial rings.

Example 2 To contrast with the previous example, consider a second little example, again with $G = \mathbb{C}^\times$, but this time taking $\gamma : z \mapsto \bar{z}^{-1}$. Then,

$$Z^1\big(\Gamma; G^\vee\big) = \big\{z \in \mathbb{C}^\times | zz^{-1} = 1\big\} = \mathbb{C}^\times,$$

and the $X_{\gamma,\chi}$ will be just $\mathbb{C}^\times$ for χ integral and empty else. The $\mathbb{C}^\times$-equivariant cohomology ring of $\mathbb{C}^\times$ is trivial; therefore the right-hand side of the conjecture is just finite-dimensional $\mathbb{C}$-vector spaces. Likewise, the left-hand side has $G(\mathbb{R}, \gamma) = S^1$, which has modules with fixed central character equivalent to the category of $\mathbb{C}$-vector spaces, too.

The overarching derived picture which interprets the conjectured equivalence via an incarnation of Koszul duality is the following: The conjecture says that the derived category $D^b(\bigoplus_\delta \mathcal{M}(G(\mathbb{R}, \delta))_\chi)$ can be embedded fully faithfully in the derived category of $\mathrm{Ext}^\bullet(\bigoplus_\pi IC_\pi)$-modules. Similarly, it is expected that the $G^\vee$-equivariant bounded constructible derived category of X, denoted $D^{bc}_{G^\vee}(X)$, can be embedded fully faithfully in the dg-derived category of dg-modules for $\mathrm{Ext}^\bullet(\bigoplus_\pi IC_\pi)$ with differential $d = 0$. We denote this category by $dgD(\mathrm{Ext}^\bullet(\bigoplus_\pi IC_\pi), d = 0)$. Very roughly, in the derived category $D(\mathrm{Ext}^\bullet(\bigoplus_\pi IC_\pi) - gr_\mathbb{Z})$ of $\mathbb{Z}$-graded $\mathrm{Ext}^\bullet(\bigoplus_\pi IC_\pi)$-modules, one can forget sort of half of the $\mathbb{Z}^2$-grading in two distinct ways:

$$\begin{array}{ccc} & D(\mathrm{Ext}^\bullet(\bigoplus_\pi IC_\pi) - gr_\mathbb{Z}) & \\ {}^{F_1}\swarrow & & \searrow^{F_2} \\ D(\mathrm{Ext}^\bullet(\bigoplus_\pi IC_\pi)) & & dgD(\mathrm{Ext}^\bullet(\bigoplus_\pi IC_\pi), d = 0). \end{array}$$

The functor F_1 forgets the imposed $\mathbb{Z}$-grading, and F_2 remembers only the total degree on our bigraded space, so we end up with a dg-module. Ultimately, the relationship described in the conjecture should be a consequence of moving from one derived category to the other through the derived category of $\mathbb{Z}$-graded $\mathrm{Ext}^\bullet(\bigoplus_\pi IC_\pi)$-modules. In the next sections, we will elaborate on the details involved in the relationships described in this introduction, give a more precise statement of the conjecture, and some indications on how to prove the known cases.

2 Equivariant Cohomology

In general, for a topological group G acting on a topological space X, we can construct $\mathrm{H}^\bullet_G(X,k)$, the G-equivariant cohomology with values in a ring k. If the G-action is topologically free, that is, if $X \to X/G$ is a principal fiber bundle, then take as the definition

$$\mathrm{H}^\bullet_G(X,k) = \mathrm{H}^\bullet(X/G,k). \tag{9}$$

In general, if the action is not topologically free, choose a G-equivariant homotopy equivalence $Y \to X$ with G acting on Y topologically freely and define

$$\mathrm{H}^\bullet_G(X,k) = \mathrm{H}^\bullet(Y/G,k). \tag{10}$$

Can we find such a G-equivariant homotopy equivalence from a space with topologically free action? If X is a point, for example, then we are looking for a contractible space such that G acts topologically freely. The Milnor construction produces such a space EG for any topological group G. Thus, $\mathrm{H}^\bullet_G(pt) = \mathrm{H}^\bullet(EG/G)$. Denote the quotient $EG/G = BG$. This is known as the classifying space for G. In the general situation, the projection $EG \times X \to X$ is a G-equivariant homotopy equivalence with respect to the diagonal action of G on $EG \times X$. Hence,

$$\mathrm{H}^\bullet_G(X) = \mathrm{H}^\bullet(EG \times_G X), \tag{11}$$

where $EG \times_G X = (EG \times X)/G$. For example, if $G = \mathbb{C}^\times$, $E\mathbb{C}^\times = \mathbb{C}^\infty \backslash \{0\}$ is contractible, and $BG = \mathbb{P}^\infty\mathbb{C}$. Then,

$$\mathrm{H}^\bullet_{\mathbb{C}^\times}(pt) = \mathrm{H}^\bullet\big(\mathbb{P}^\infty\mathbb{C}\big) = k[t] \tag{12}$$

with $t \in \mathrm{H}^2(\mathbb{P}^\infty\mathbb{C};k)$ any generator of this k-module.

Next, we want to define the equivariant derived category for a topological space X with the action of a topological group, following [BL94]. One can associate to X the abelian category $\mathcal{A}b_X$ of sheaves of abelian groups on X. Let $D(\mathcal{A}b_X) = D(X)$ denote the corresponding derived category. In $\mathcal{A}b_X$ there is a special sheaf, which we will write as $\mathbb{Z}_X$ or $\underline{X}$, that is constant with coefficients in $\mathbb{Z}$. Then, the sheaf cohomology of $\mathbb{Z}_X$ is

$$\mathrm{H}^i(X,\mathbb{Z}_X) = \mathrm{Hom}_{D(X)}\big(\underline{X},\underline{X}[i]\big) = \mathrm{Ext}^i_{\mathcal{A}b_X}(\underline{X},\underline{X}), \tag{13}$$

which leads to a ring isomorphism $\mathrm{H}^\bullet(X,\mathbb{Z}_X) \simeq \mathrm{Ext}^\bullet_{\mathcal{A}b_X}(\underline{X},\underline{X})$.

Ideally, the same type of equality should hold for the correct definition of the G-equivariant derived category $D_G(X)$. Instead of $\mathcal{A}b_X$, we begin with the abelian category of G-equivariant abelian sheaves. One way to define equivariant sheaves is via the formalism of étale spaces. Recall that a sheaf of sets $\mathscr{F}$ on X is the same as an étale map $\bar{\mathscr{F}} \to X$ from the étale space for $\mathscr{F}$ to X. In other words, the category of sheaves of sets on X is equivalent to the category of étale maps to X. A sheaf $\mathscr{F}$ on X is said to be G-equivariant if there exists a G-action on $\bar{\mathscr{F}}$ so that the map $\bar{\mathscr{F}} \to X$ is G-equivariant.

Example Let $\mathbb{C}^\times$ act on a point. Every sheaf on a point is $\mathbb{C}^\times$-equivariant for the trivial action, and in fact forgetting the action gives an equivalence of categories from equivariant to nonequivariant sheaves in this case. Therefore, the derived category of equivariant sheaves is equal to the derived category of abelian sheaves, and therefore the corresponding equivariant cohomology should equal the regular cohomology, right? However, we saw already that this is not the case, so for the equivariant derived category, we cannot simply take the derived category of the category of equivariant abelian sheaves.

The correct construction of the equivariant derived category is less obvious. Take the diagram

$$\begin{array}{ccc} & EG \times X & \\ {}^{q}\swarrow & & \searrow^{p} \\ EG \times_G X & & X \end{array} \tag{14}$$

The G-equivariant derived category $D_G(X)$ of sheaves on X has as its objects

$$D_G(X) = \left\{\mathscr{F}^\bullet \in D(EG \times_G X) | \exists \mathscr{G}^\bullet \in D(X) \text{ such that } p^*\mathscr{G}^\bullet \simeq q^*\mathscr{F}^\bullet\right\}. \tag{15}$$

In other words, G-equivariant complexes of sheaves on X are complexes of sheaves on X which, when pulled back to $EG \times X$, are quasi-isomorphic to complexes of sheaves pulled back from X. One trick involved in this definition is the utilization of the fact that since EG is contractible, p^* is fully faithful, so we can consider $D(X)$ to be a full triangulated subcategory of $D(EG \times X)$.

Returning to our previous example of $\mathbb{C}^\times$ acting on a point, we will compute $D_{\mathbb{C}^\times}(pt)$ explicitly. In this case, (14) becomes

$$\begin{array}{ccc} & \mathbb{C}^\infty \backslash \{0\} & \\ {}^{q}\swarrow & & \searrow^{p} \\ \mathbb{P}^\infty\mathbb{C} & & pt \end{array} \tag{16}$$

Then, $D_{\mathbb{C}^\times}(pt)$ has as objects complexes of sheaves $\mathscr{F}^\bullet$ in $D(\mathbb{P}^\infty\mathbb{C})$ with constant cohomology sheaves. This is a nice description, but we can do even better. By imposing some finiteness conditions such as boundedness (b) and forcing the complexes to have constant cohomology of finite dimension (c), we find, say in the case of complex coefficients $k = \mathbb{C}$, that there is a fully faithful embedding

$$D^{bc}_{\mathbb{C}^\times}(pt) \subset dgD\big(\mathbb{C}[t]\big), \tag{17}$$

where $dgD(\mathbb{C}[t])$ is the differential graded derived category of $\mathbb{C}[t]$-modules, with t a degree 2 element. The embedding is generated by the map $\underline{pt} \mapsto \mathbb{C}[t]$.

The construction of $dgD(\mathbb{C}[t])$ is slightly different from the construction of a derived category from an abelian category, essentially since we can think of $\mathbb{C}[t]$-modules as complexes themselves. We say that a ring A is a differential graded (dg) ring if $A = \bigoplus_{i\in\mathbb{Z}} A^i$, there is a unit $1 \in A^0$, and there is a degree one map $d : A^i \to A^{i+1}$ satisfying the Leibniz rule on homogeneous elements for $a \in A^i$,

$$d(ab) = (da)b + (-1)^i a(db). \tag{18}$$

A dg A-module is a $\mathbb{Z}$-graded module $M = \bigoplus_{i\in\mathbb{Z}} M^i$ with grading respected by the A-action,

$$A^i \times M^j \to M^{i+j},$$

together with a differential such that on homogeneous elements $a \in A^i$, $m \in M^j$,

$$d(am) = (da)m + (-1)^i a(dm). \tag{19}$$

For a given dg A-module M and any integer $k \in \mathbb{Z}$, we can define another dg A-module $M[k]$, called the shift of M. The shift $M[k]$ has the ith graded component

$$M[k]^i = M^{i+k}$$

and the ith differential $d^i_{M[k]} = (-1)^k d^{i+k}_M$. With shifts, we define the notion of degree k homomorphisms of the underlying graded modules

$$\mathrm{Hom}^k_A(N, M) = \mathrm{Hom}_{gr}\big(N, M[k]\big) = \prod_{i\in\mathbb{Z}} \mathrm{Hom}\big(N^i, M^{i+k}\big)$$

for dg-modules N and M, where the central Hom space is in the category of graded modules for the underlying graded ring of A. There is a natural differential

$$d^k : \mathrm{Hom}^k(N, M) \to \mathrm{Hom}^{k+1}(N, M)$$

taking a degree k morphism f to $d^k(f)$, which in degree i equals

$$d^k(f)^i = d^{i+k}_M \circ f^i - (-1)^k f^{i+1} \circ d^i_N.$$

Then, $\mathrm{Hom}^0(N, M) = \mathrm{Hom}_{dg}(N, M)$, homomorphisms in the category of dg-modules for A, and we define the homotopy category of dg A-modules, denoted $dgHot_A$, to have dg A-modules as objects, and $\mathrm{H}^0(\mathrm{Hom}^\bullet(N, M))$, the zeroth cohomology of the Hom complex, as morphisms between objects N and M. The zeroth cohomology of the Hom complex corresponds to homotopy equivalence classes of A-equivariant morphisms of dg-modules. The dg homotopy category $dgHot_A$ is a triangulated category. Then, we define the dg derived category of A-modules $dgD(A)$ to be the localization of $dgHot_A$ with respect to quasi-isomorphisms.

Why should dg-categories come in? In the setting of abelian categories, the Freyd–Mitchell theorem approximately states that every abelian category is equivalent to a category of modules over some ring. This is not true without some finiteness conditions, but it can basically be used at will to prove many theorems about arbitrary abelian categories. The corresponding moral for derived categories is that any "good" derived category should be equivalent to $dgD(A)$ for some dg ring A. The derived version of the conjecture given in the introduction then would be an example of this heuristic. Assuming that the Harish-Chandra categories satisfy the conditions of "good," we have left to justify the extension ring taking the role of A.

3 Tilting Complexes

Why should modules over a self-extension ring appear in the conjecture? First, we should be a bit more explicit about what the self-extension ring is. Let $\mathcal{I}$ be an additive category, and $\mathit{Kom}(\mathcal{I})$ the category of complexes of objects in $\mathcal{I}$. The Hom-complex defined in the previous section did not depend on the objects being modules. Consequently, for any category of complexes such as $\mathit{Kom}(\mathcal{I})$, we can define the Hom-complex $\mathrm{Hom}^\bullet(T, T')$ between two objects T and T' of $\mathit{Kom}(\mathcal{I})$ as above. Then, the complex of self-extensions of an object T is

$$\mathrm{Ext}^\bullet_{\mathcal{I}}(T) = \mathrm{Hom}^\bullet(T, T).$$

Proposition 1 *For $\mathcal{I}$ additive, $\mathit{Kom}(\mathcal{I})$ its category of complexes, and T an object of $\mathit{Kom}(\mathcal{I})$, the self-extension complex $E = \mathrm{Ext}^\bullet_{\mathcal{I}}(T)$ of T has the structure of a dg-ring.*

If $\mathit{Hot}(\mathcal{I})$ denotes the usual homotopy category of $\mathcal{I}$, the object T generates a full triangulated subcategory $\langle T\rangle_\Delta \subset \mathit{Hot}(\mathcal{I})$, formed by closing the subcategory consisting of the single object T, with all endomorphisms in $\mathcal{I}$, under triangulated category operations. It can be seen trivially that there is an equivalence of triangulated categories

$$\langle T\rangle_\Delta \xrightarrow[\mathrm{Hom}^\bullet_{\mathcal{I}}(T,-)]{\sim} \mathit{dgfree}\text{-}E \tag{20}$$

between $\langle T\rangle_\Delta$ and the category of free finite-rank dg E-modules with homotopy equivalence classes of *dg*-morphisms. Free finite-rank dg E-modules are here understood to be the triangulated subcategory of dgHot_E generated by E. In other words,

$$\mathit{dgfree}\text{-}E := \langle E\rangle_\Delta.$$

Clearly, the functor $\mathrm{Hom}^\bullet_{\mathcal{I}}(T, -) : \mathit{Hot}(\mathcal{I}) \to \mathit{dgHot}_E$ and sends $T \mapsto E$. Moreover, this functor is fully faithful, more-or-less by definition.

The corresponding derived picture is somewhat more complicated. Let $\mathcal{A}$ be an abelian category, $\mathit{Kom}(\mathcal{A})$ again its category of complexes, and $\mathit{Hot}(\mathcal{A})$, $D(\mathcal{A})$ its homotopy and derived categories, respectively.

Definition 1 An object $T \in Kom(\mathcal{A})$ is *endacyclic* if $Hot_{\mathcal{A}}(T, T[n]) \simeq D_{\mathcal{A}}(T, T[n])$ for all n.

In $\mathcal{A}$, a *tilting object* T is an object which has no higher self-extensions, so an object which is endacyclic when understood as a complex.

Proposition 2 *Let $\mathcal{A}$ be an abelian category, and $T \in Kom(\mathcal{A})$ an endacyclic (aka tilting) complex. If $E = \mathrm{Ext}^{\bullet}_{\mathcal{A}}(T)$, then*

$$\langle T \rangle_{\Delta} \xrightarrow[\mathrm{Hom}^{\bullet}_{\mathcal{A}}(T,-)]{\sim} dgfree\text{-}E \tag{21}$$

as triangulated subcategories of $D(\mathcal{A})$ and $dgD(E)$, respectively.

In fact, in this proposition, E is an endacyclic object in $dgHot(E)$, so we actually have $\langle E \rangle_{\Delta} = \langle E \rangle_{\Delta}$, where the left-hand side is to be understood in $dgHot(E)$, and the right-hand side in $dgD(E)$. Likewise, T tilting implies that $\langle T \rangle_{\Delta}$ is the same triangulated category whether constructed in $Hot(\mathcal{A})$ or $D(\mathcal{A})$, but we do not necessarily have $Hot(\mathcal{A}) \simeq D(\mathcal{A})$. The proof of this second proposition is thus essentially the same as for the first proposition—insisting on T tilting is the only difference.

Next, we will try to understand $D_T(pt)$ for T a torus. We had before that $D_{\mathbb{C}^{\times}}(pt)$ is the category of complexes of sheaves $\mathscr{F}^{\bullet}$ in $D(\mathbb{P}^{\infty}\mathbb{C})$ with constant cohomology. Inside $D_{\mathbb{C}^{\times}}(pt)$ is $D^{bc}_{\mathbb{C}^{\times}}(pt)$, the subcategory consisting of complexes with bounded finite-rank cohomology sheaves. In fact,

$$D^{bc}_{\mathbb{C}^{\times}}(pt) = \langle \underline{pt} \rangle_{\Delta} = \left\langle \underline{\mathbb{P}^{\infty}\mathbb{C}} \right\rangle_{\Delta}. \tag{22}$$

Now we want to apply the isomorphism in Proposition 2. Let $\mathcal{A}$ be the abelian category of sheaves of $\mathbb{C}$-vector spaces on $\mathbb{P}^{\infty}\mathbb{C}$. Unfortunately, the constant sheaf $\underline{\mathbb{P}^{\infty}\mathbb{C}}$ is not endacyclic. We must replace this by a quasi-isomorphic object which *is* endacyclic. Choose and injective resolution $\underline{\mathbb{P}^{\infty}\mathbb{C}} \tilde{\to} \mathscr{I}^{\bullet}$. Then,

$$\left\langle \underline{\mathbb{P}^{\infty}\mathbb{C}} \right\rangle_{\Delta} \simeq dgfree\text{-}\mathrm{Ext}^{\bullet}_{\mathcal{A}}(\mathscr{I}^{\bullet}). \tag{23}$$

Let $E = \mathrm{Ext}^{\bullet}_{\mathcal{A}}(\mathscr{I}^{\bullet})$. Unfortunately, $\mathscr{I}^{\bullet}$ can be supported in many, even infinitely many, nonnegative degrees, and E might even live in all degrees, alias be unbounded in both directions and could be very big, so it will be uncomfortable to consider *dgfree-E*.

We need to understand a bit more about differential graded algebras. If we have a usual ring A, consider the category A-mod of left A-modules, and similarly for a ring B, consider mod-B, the category of right B-modules. Then, take X to be an A–B-bimodule. There is a pair of adjoint functors

$$mod - B \underset{\mathrm{Hom}_A(X,-)}{\overset{-\otimes_B X}{\rightleftarrows}} A - mod. \tag{24}$$

The same constructions more-or-less carry over to *dg*-algebras. Let B, A be *dg*-algebras, and X a A–B-bimodule. Then, $dgD(A)$ is the dg derived category of left A-modules, as before, and we use $dgD(B)^R$ to denote the dg derived category of right B-modules. The derived functors

$$dgD(B)^R \underset{\mathrm{RHom}_A(X,-)}{\overset{-\otimes^L_B X}{\rightleftarrows}} dgD(A) \tag{25}$$

are again an adjoint pair. If X has the property that there exists $c \in \mathrm{H}^0 X$ such that $\mathrm{H}^0 X$ as an $\mathrm{H}^0 A$-$\mathrm{H}^0 B$-bimodule is free with basis c, then A-*dgfree* $\simeq$ B-*dgfree* via the above adjoint pair of functors. That is to say, provided that a bimodule such as X exists, we can avoid working with modules over E by moving to the *dgfree* category for a simpler algebra. Specifically, in the example of $\mathbb{C}^\times$-equivariant sheaves on a point, if we let $t \in \mathrm{H}^2_{\mathbb{C}^\times}(pt)$ be a generator of the equivariant cohomology ring, $\mathrm{H}^\bullet_{\mathbb{C}^\times}(pt) = \mathbb{C}[t]$ is a dga with $d = 0$. Moreover, the natural inclusion $\mathbb{C}[t] \to E$ is a quasi-isomorphism. This is a special case of the equivalence above, with $A = X = E$ and $B = \mathbb{C}[t]$. Therefore,

$$dgfree\text{-}E \simeq dgfree\text{-}\mathbb{C}[t].$$

The reader familiar with A_∞-algebras will see the A_∞ structure underlying E is trivial. Roughly, the A_∞ interpretation of these statements is that if the canonical A_∞ structure on $\mathrm{H}^\bullet A = \mathrm{H}^\bullet B$ for A, B dga's, then their derived categories agree.

Instead of $\mathbb{P}^\infty\mathbb{C}$, we can try to apply these constructions to other manifolds, such as Kähler manifolds. If X is a paracompact C^∞-manifold and $D(X, \mathbb{R})$ is the derived category of abelian sheaves on X with $\mathbb{R}$-coefficients, then

$$D(X, \mathbb{R}) \supset \langle \underline{X} \rangle_\Delta \simeq dgfree\text{-}\Omega^\bullet_X, \tag{26}$$

where $\Omega^\bullet_X$ is the deRham complex for X. Now, if X is compact and Kähler, then we have in addition

$$dgfree\text{-}\Omega^\bullet_X \simeq dgfree\text{-}\mathrm{H}^\bullet(\underline{X}).$$

This isomorphism implies that $\mathrm{H}^\bullet(\underline{X})$ controls most of the homotopy of X. The trick in the proof of this isomorphism is to do some splitting by Hodge theory, see [DGMS75].

At this point, we return to the conjecture of the introduction. Given a connected reductive algebraic group G over $\mathbb{C}$, an anti-holomorphic involution γ on G, a character χ of Z, the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G, we formed for the dual group $G^\vee$ (on which γ acts holomorphically), a space $X_{\gamma,\chi} = G^\vee \times_{P^\vee(\chi)} Z^1(\Gamma, G^\vee)$. Then:

Conjecture 1

$$\bigoplus_{\delta\in \mathrm{H}^1(\Gamma,G)} \mathcal{M}\big(G(\mathbb{R},\delta)\big)_\chi \simeq \mathrm{Ext}^\bullet_{G^\vee}\Big(\bigoplus_{\pi\in \mathrm{Par}(X)} IC_\pi\Big)\text{-}\mathcal{N}il. \tag{27}$$

The point I want to make is that the extension algebra on the right should be quasi-isomorphic to the *dg*-algebra it was born from. For such a conjecture to make sense, very much in the way we saw above in simple cases. Up until now, we have treated the cases $G^\gamma = \mathbb{R}^\times$ and $G^\gamma = S^1$ in the examples of the introduction. Next, we will approach $SL(2,\mathbb{R})$.

We should emphasize that this approach is in some sense orthogonal to localization, and in some sense "better," as is works quite smoothly for singular central character, p-adic fields, Z not acting by characters, etc. However, the problem is that it is but a conjecture.

4 The Case of $SL(2,\mathbb{R})$

In this lecture, we will look at the conjecture for $SL(2,\mathbb{R})$. Let $G = SL(2,\mathbb{C})$, and γ usual complex conjugation. Then, $\mathrm{H}^1(\Gamma; G)$ has a single element, so the sum

$$\bigoplus_{\delta\in \mathrm{H}^1(\Gamma;G)} \mathcal{M}\big(G(\mathbb{R},\delta)\big)_\chi = \mathcal{M}\big(SL(2,\mathbb{R})\big)_\chi. \tag{28}$$

On the other side, $G^\vee = PSL(2,\mathbb{C})$, and γ acts as the identity. Hence,

$$Z^1\big(\Gamma; G^\vee\big) = \big\{g\in G^\vee | g^2 = 1\big\}.$$

The identity e, of course, is in this set, and any nontrivial $g \in Z^1(\Gamma; G^\vee)$ is semi-simple, therefore contained in a unique torus $T^\vee$. Let $N^\vee$ be the normalizer of $T^\vee$. Then the quotient $G^\vee/N^\vee$ parameterizes all other nontrivial elements of $Z^1(\Gamma; G^\vee)$. In this way, we see

$$Z^1\big(\Gamma; G^\vee\big) = \{e\} \sqcup G^\vee/N^\vee.$$

Returning to $\mathcal{M}(SL(2,\mathbb{R}))_\chi$, choose a basis $\{X, H, Y\}$ of $\mathfrak{sl}(2,\mathbb{C})$ in such a way that

$$\begin{aligned} \langle H\rangle_{\mathbb{C}} &= \mathfrak{so}(2,\mathbb{C}) \subset \mathfrak{sl}(2,\mathbb{C}),\\ [X,Y] &= H,\\ [H,X] &= 2X, \quad \text{and}\\ [H,Y] &= -2Y. \end{aligned} \tag{29}$$

In other words, the usual relations for a basis of $\mathfrak{sl}(2,\mathbb{C})$ are satisfied. Then, the category of Harish-Chandra modules for $SL(2,\mathbb{R})$ breaks up into two distinct pieces:

$$\mathcal{M}\big(SL(2,\mathbb{R})\big) = \mathcal{M}^{\mathrm{ev}} \oplus \mathcal{M}^{\mathrm{odd}}, \tag{30}$$

where all eigenvalues of H are even or odd integers, respectively. These representations look like $M = \bigoplus_{i\in\mathbb{Z}} M_{2i}$ if M is even, with M_{2i} the $2i$ eigenspace of H, and the action of X and Y visualized by the diagram

$$\dots M_{-2} \underset{Y}{\overset{X}{\rightleftarrows}} M_0 \underset{Y}{\overset{X}{\rightleftarrows}} M_2 \dots \tag{31}$$

The category of even representations with respect to the most singular central character χ can be seen to be isomorphic to

$$\mathcal{M}_\chi^{\mathrm{ev}} \tilde{\to} \hat{Z}_\chi\text{-}mod_{fd} \tag{32}$$

with the isomorphism taking M to M_0, its zero H-eigenspace. Here, $\hat{Z}_\chi$ is the completion of Z with respect to the maximal ideal determined by χ. In this case, the equivalence indicates that the action of Z on M_0 controls the entire module M. For the modules with odd eigenspaces,

$$\mathcal{M}_\chi^{\mathrm{odd}} \tilde{\to} Rep^{fd}\left(\bullet \underset{\psi}{\overset{\phi}{\rightleftarrows}} \bullet \,\middle|\, \phi\psi \text{ nilpotent} \right). \tag{33}$$

That is, odd representations are the same as finite-dimensional quiver representations for the quiver above, with $\phi\psi$ acting nilpotently. In this case, the functor is given by sending $M = \bigoplus_{i\in\mathbb{Z}} M_{2i+1}$ to the quiver representation

$$M_{-1} \underset{\psi=Y}{\overset{\phi=X}{\rightleftarrows}} M_1.$$

The nilpotence arises from the fact that $\Omega + 1$ acts nilpotently on M, where Ω is the Casimir operator.

Let us now try to understand the geometric extension algebra and see if we can decompose it into these same pieces. Recall $Z^1(\Gamma, G^\vee) = pt \sqcup G^\vee/N^\vee$. For a complex algebraic group G acting on a complex algebraic variety X with finitely many orbits, we defined $\mathrm{Par}(X)$ to be the set of pairs (Y, τ) consisting of a G-orbit Y and a G-equivariant irreducible local system τ; equivalently, an irreducible complex representation of the component group of the stabilizer G_y of some fixed point $y \in Y$. From this data we can form, for fixed parameter $\pi = (Y, \tau)$, the irreducible object $\mathscr{L}^\pi = IC_\pi \in D_G(X)$. Then, build the equivariant self-extension complex

$$\mathrm{Ext}_G^\bullet(X) = \bigoplus_{i\in\mathbb{N}} \mathrm{Hom}_{D_G(X)}\left(\bigoplus_\pi \mathscr{L}^\pi, \bigoplus_\pi \mathscr{L}^\pi[i] \right). \tag{34}$$

We need to understand IC_π better in order to go any further. Recall

$$X = X_{\gamma,\chi} = G^\vee \times_{P^\vee(\chi)} \left(pt \sqcup G^\vee/N^\vee\right).$$

Each of pt and $G^\vee/N^\vee$ is a distinct $G^\vee$ orbit. If χ is maximally singular, then $P^\vee(\chi) = G^\vee$ and

$$\mathrm{Ext}^\bullet_{G^\vee}(\underline{pt}) = \mathrm{H}^\bullet_{G^\vee}(pt) = \mathcal{O}^{\mathcal{W}}_{Lie(T^\vee)}, \tag{35}$$

where $T^\vee$ is a maximal torus in $G^\vee$, and $\mathcal{W}$ is the Weyl group of $G^\vee$. By $\mathcal{O}_{Lie(T^\vee)}$ we mean the ring of regular algebraic functions on $Lie(T^\vee)$. This ring is the same as $S(Lie(T))$, the symmetric algebra over the Lie algebra $Lie(T)$, which is dual to $Lie(T^\vee)$. Recall also that Harish-Chandra's isomorphism further identifies $Z \tilde{\to} S(Lie(T))^{\mathcal{W}}$. Under this isomorphism, the most singular character χ corresponds to the 0 character of $S(Lie(T))^{\mathcal{W}}$. Then, the even part of $\mathcal{M}_\chi$ corresponds to the cohomology of the point:

$$\mathcal{M}^{\mathrm{ev}}_\chi \tilde{\to} \hat{Z}_\chi\text{-}mod_{fd} = \mathrm{H}^\bullet_{G^\vee}(pt)\text{-}\mathcal{N}il \leftrightarrow \hat{S}\left(Lie(T)\right)^{\mathcal{W}}_0\text{-}\mathcal{N}il, \tag{36}$$

where as before, $\hat{Z}_\chi$ denotes completion with respect to χ, and similarly for $\hat{S}(Lie(T))^{\mathcal{W}}_0$. Now, the odd part must correspond to $G^\vee/N^\vee$. How do we understand the self-extension algebra in this case? There is a general equivalence of categories for a closed subgroup $H \subset G$,

$$D_G(G \times_H Y) \simeq D_H(Y). \tag{37}$$

Likewise, $\mathrm{Ext}^\bullet_G(G \times_H Y) \simeq \mathrm{Ext}^\bullet_H(Y)$. This means, in particular, that we can identify

$$\mathrm{Ext}^\bullet_{G^\vee}\left(G^\vee/N^\vee\right) = \mathrm{Ext}^\bullet_{N^\vee}(pt)$$

for the purposes of calculation.

Lemma 1 *Let N be a complex algebraic group, and N° its identity component with $\mathcal{W} = N/N^\circ$ a finite abelian group. Then,*

$$\mathrm{Ext}^\bullet_N(pt) = \mathrm{Ext}^\bullet_{N^\circ}(pt)^\sigma[\mathcal{W}] = \mathrm{H}^\bullet_{N^\circ}(pt)^\sigma[\mathcal{W}]. \tag{38}$$

The middle term $\mathrm{Ext}^\bullet_{N^\circ}(pt)^\sigma[\mathcal{W}]$ is the twisted group ring of $\mathcal{W}$, defined in general as follows. For a ring R and a group $\mathcal{W}$ acting on R by σ, $R^\sigma[\mathcal{W}]$ is the twisted group ring of $\mathcal{W}$ over R with

$$rw = wr^{\sigma(w)} \quad \forall r \in R,\ w \in \mathcal{W}. \tag{39}$$

In the context of the lemma, N acts by conjugation on N° and therefore on $\mathrm{Ext}^\bullet_{N^\circ}(pt)$, but N° fixes everything in this ring, so the action descends to an action σ of $\mathcal{W}$.

In the $SL(2,\mathbb{R})$ example, the lemma implies

$$\mathrm{Ext}^\bullet_{G^\vee}\left(G^\vee/N^\vee\right) = \mathrm{H}^\bullet_{T^\vee}(pt)^\sigma[\mathcal{W}],$$

since $\mathcal{W} = \{\pm 1\}$ is abelian.

Exercise Check that $\mathrm{H}^{\bullet}_{T^{\vee}}(pt)^{\sigma}[\mathcal{W}]$ is really the quiver algebra of the quiver

$$\bullet \underset{\psi}{\overset{\phi}{\rightleftarrows}} \bullet \tag{40}$$

with $\phi\psi$ nilpotent.

To contrast with this most singular character case, take instead χ regular. If χ is generic, it satisfies no integrality properties. Without integrality of χ, irreducible Harish-Chandra modules are all principal series, and these admit only self-extensions. These are described by Zuckerman. On the other side, $X_{\gamma,\chi}$ turns out to be the disjoint union of 2^n copies of $G^{\vee}/T^{\vee}$, with n equal to the rank of $T^{\vee}$. There are 2^n-many principal series. So, generically, everything is well understood and easily computed on both sides of the conjecture.

5 $SL(2,\mathbb{R})$ Continued, χ an Integral Character

As in the last section, let G be a connected reductive algebraic group, and γ a quasi-split antiholomorphic involution. To these data, fix $\chi \in CharZ$. Then, recall:

Conjecture

$$\bigoplus_{\delta \in \mathrm{H}^1(\Gamma;G)} \mathcal{M}\big(G(\mathbb{R},\delta)\big)_{\chi} \simeq \mathrm{Ext}^{\bullet}_{G^{\vee}}(X_{\gamma,\chi})\text{-}\mathcal{N}il. \tag{41}$$

The right-hand side is the category of finite-dimensional modules for $\mathrm{Ext}^{\bullet}_{G^{\vee}}(X_{\gamma,\chi})$ which are annihilated by high degrees, and for integral χ, we have $X_{\gamma,\chi} = G^{\vee} \times_{P^{\vee}(\chi)} Z^1(\Gamma;G)$, where $P^{\vee}(\chi)$ is a parabolic of $G^{\vee}$ determined by χ. We should explain a few more details about this construction. The cocycles $Z^1(\Gamma;G^{\vee})$ were identified as

$$Z^1\big(\Gamma;G^{\vee}\big) = \big\{g \in G \,|\, gg^{\gamma} = 1\big\}, \tag{42}$$

but what is the action of Γ on $G^{\vee}$? Choose in G a Borel and maximal torus $G \supset B \supset T$ stable under Γ. Then, this data determines $\mathcal{X}(T) \supset R \supset R^{+}$, the character lattice of T, its root system R, and positive roots R^{+}, upon which γ acts by $\lambda^{\gamma} = \overline{\lambda \circ \gamma}$ for $\lambda \in \mathcal{X}(T)$. This makes the diagram

$$\begin{array}{ccc} T & \xrightarrow{\lambda^{\gamma}} & \mathbb{C}^{\times} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\bar{z}} \\ T & \xrightarrow{\lambda} & \mathbb{C}^{\times} \end{array} \tag{43}$$

commute, with $\bar{z}$ indicating complex conjugation on $\mathbb{C}^{\times}$.

One can dualize the data $\mathcal{X}(T) \supset R \supset R^+$ by simply taking the colattice with coroots

$$\mathcal{X}^*(T) \supset R^\vee \supset \left(R^+\right)^\vee,$$

determining now the action of γ on $G^\vee$ as a holomorphic involution.

The parabolic $P^\vee(\chi)$ is the parabolic subgroup of $G^\vee$ given by the simple roots corresponding to the walls on which χ lies. So, for regular χ, the Borel dual $P^\vee(\chi) = B^\vee$ to B, and for the most singular character χ, we have $P^\vee(\chi) = G^\vee$. These are the two extremes, and all other cases lay between. In our usual notation, for a complex algebraic group G acting on a complex algebraic variety X, define the graded ring $\mathrm{Ext}^\bullet_G(X)$ to be

$$\mathrm{Ext}^\bullet_G(X) := \mathrm{Ext}^\bullet_G\Bigg(\bigoplus_{\pi \in \mathrm{Par}(X)} IC_\pi\Bigg). \tag{44}$$

With respect to the $SL(2,\mathbb{R})$ example, there is a second real form of $SL(2,\mathbb{C})$ that we have not considered, $SU(2)$. For regular χ,

$$G^\vee \times_{G^\vee} G^\vee/N^\vee = G^\vee/B^\vee \times G^\vee/N^\vee = G^\vee \times_{N^\vee} G^\vee/B^\vee.$$

Here we see the involvement of $N^\vee$ orbits on $G^\vee/B^\vee$, the flag variety of $PGL(2)$. Since $SL(2,\mathbb{C})$ is a double cover of $PGL(2)$, this allows for locally constant sheaves with monodromy which did not appear for $PGL(2)$. Such local systems correspond to representations of $SU(2)$.

The conjecture for $SL(2,\mathbb{R})$ at the trivial character, stated as

$$\mathcal{M}\big(SL(2,\mathbb{R})\big)_{\mathrm{triv}} \simeq \mathrm{Ext}^\bullet_{PGL(2,\mathbb{C})}\big(pt \sqcup G^\vee/N^\vee\big)\text{-}\mathcal{N}il, \tag{45}$$

ignores all compact forms of $SL(2,\mathbb{C})$. In general, the corresponding statement which includes compact forms is not nice to write down, but it happens to be reasonable in this example. If we include the compact forms of $SL(2,\mathbb{C})$ in the conjecture, it becomes

$$\begin{aligned}\mathcal{M}\big(SU(1,1)\big)_{\mathrm{triv}} \oplus \mathcal{M}\big(SU(0,2)\big)_{\mathrm{triv}} \oplus \mathcal{M}\big(SU(2,0)\big)_{\mathrm{triv}} \\ \simeq \mathrm{Ext}^\bullet_{SL_2(\mathbb{C})}\big(pt \sqcup G^\vee/N^\vee\big)\text{-}\mathcal{N}il. \end{aligned}\tag{46}$$

Note that the right-hand side differs from the previous statement of the conjecture, since we now look at $SL(2,\mathbb{C})$, the double cover of $PGL(2,\mathbb{C})$. In fact, it can be seen that

$$\mathrm{Ext}^\bullet_{SL_2(\mathbb{C})}\big(pt \sqcup G^\vee/N^\vee\big) = \mathbb{C}^2 \times \mathrm{Ext}^\bullet_{PGL(2,\mathbb{C})}\big(pt \sqcup G^\vee/N^\vee\big), \tag{47}$$

with each contribution of $\mathbb{C}$ corresponding to a compact form.

As χ becomes more singular, $P^\vee(\chi)$ grows larger, and we have a natural projection $X_{\gamma,\chi} \to X_{\gamma,\chi'}$ for $P^\vee(\chi') \supset P^\vee(\chi)$. This corresponds to translation to the walls of Harish-Chandra modules.

How does Z act on the right-hand side of the conjecture? There is a projection $G^\vee \times_{P^\vee(\chi)} Z^1(\Gamma; G)$ to the partial flag variety $G^\vee/P^\vee(\chi)$ which induces the inclusion

$$\mathcal{O}_{Lie(T^\vee)}^{\mathcal{W}_{P^\vee(\chi)}} = \mathrm{H}^\bullet_{P^\vee(\chi)}(pt) = \mathrm{H}^\bullet_{G^\vee}\big(G^\vee/P^\vee(\chi)\big) \hookrightarrow \mathrm{Ext}^\bullet_{G^\vee}(X_{\gamma,\chi}), \tag{48}$$

where $\mathcal{W}_{P^\vee(\chi)}$ is the Weyl group of the Levi factor of $P^\vee(\chi)$. Also, the completion of Z at χ, $\hat{Z}_\chi$, should be isomorphic to the completion of $\mathcal{O}_{Lie(T^\vee)}^{\mathcal{W}_{P^\vee(\chi)}}$ along its grading. Why is that? We know that $\hat{Z}_{\mathrm{triv}} \simeq \hat{S}(Lie(T))$, the completion of the symmetric algebra of $Lie(T)$. At the most singular character, $\hat{Z}_{\mathrm{sing}} \simeq \hat{S}^{\mathcal{W}}$, so in this case we do see only the Weyl group invariants.

Now, concentrate for a moment on $\mathcal{M}(SL(2,\mathbb{R}))_{\mathrm{triv}}$. We still have the decomposition

$$\mathcal{M}_{\mathrm{triv}} = \mathcal{M}_{\mathrm{triv}}^{\mathrm{ev}} \oplus \mathcal{M}_{\mathrm{triv}}^{\mathrm{odd}}.$$

The odd component corresponds to the point in $Z^1(\Gamma; G) = pt \sqcup G^\vee/N^\vee$, as before. The even piece can be identified with quiver representations for the quiver with relations

$$\bullet \underset{\psi_0}{\overset{\phi_0}{\rightleftarrows}} \bullet \underset{\psi_1}{\overset{\phi_1}{\rightleftarrows}} \bullet, \qquad \phi_0\psi_0 = \psi_1\phi_1, \tag{49}$$

and with $\psi_0\phi_0$, $\phi_1\psi_1$ acting nilpotently. The functor sends $M = \bigoplus_{i\in\mathbb{Z}} M_{2i}$ to

$$M_{-2} \underset{Y}{\overset{X}{\rightleftarrows}} M_0 \underset{Y}{\overset{X}{\rightleftarrows}} M_2. \tag{50}$$

We need to see that the quiver algebra for this quiver is equal to $\mathrm{Ext}^\bullet_{PGL(2,\mathbb{C})}(G^\vee \times_{B^\vee} G^\vee/N^\vee)$. We can interpret quiver representations for (49) in terms of equivariant sheaves on the flag variety $\mathbb{P}^1$ for $G^\vee$. The $T^\vee$ orbits on $\mathbb{P}^1$ are $\mathbb{P}^1 = \{0\} \sqcup \mathbb{C}^\times \sqcup \{\infty\}$, and its normalizer $N^\vee$ identifies 0 and ∞, so $\mathbb{P}^1$ has $B^\vee$ orbits $\mathbb{P}^1 = \{0,\infty\} \sqcup \mathbb{C}^\times$. There is a unique (trivial) local system $\underline{Y}$ on the orbit $Y = \{0,\infty\}$. On $X = \mathbb{C}^\times$, there is one local system $\underline{X}^\pm$ for each irreducible representation of the isotropy group of a point $x \in X$. Therefore,

$$\mathrm{Ext}^\bullet_{N^\vee}\big(\mathbb{P}^1\big) = \mathrm{Ext}^\bullet_{N^\vee}\big(\underline{X}^+[1] \oplus \underline{X}^-[1] \oplus \underline{Y}\big), \tag{51}$$

where we abuse the notation by denoting the local system by the same name as its direct image under the inclusion of the corresponding orbit. Also, the shift by the complex dimension of the orbit must be introduced for Poincaré duality to occur in the hypercohomology of the local system. The direct image of $\underline{Y}$ corresponds to the trivial representation, and its self-extensions correspond to local cohomology about points. The $\underline{X}^{\pm}$, on the other-hand, correspond to discrete series. This is the opposite of what happens in localization. Also, on X, there are two local systems which do not extend to all of $\mathbb{P}^1$. These correspond to $SU(2,0)$ and $SU(0,2)$. The equalities

$$\begin{aligned}\mathrm{Ext}^{\bullet}_{PGL(2,\mathbb{C})}\left(G^{\vee}\times_{B^{\vee}}G^{\vee}/N^{\vee}\right) &= \mathrm{Ext}^{\bullet}_{PGL(2,\mathbb{C})}\left(G^{\vee}\times_{N^{\vee}}G^{\vee}/B^{\vee}\right)\\ &= \mathrm{Ext}^{\bullet}_{N^{\vee}}\left(G^{\vee}/B^{\vee}\right)\\ &= \mathrm{Ext}^{\bullet}_{N^{\vee}}\left(\mathbb{P}^1\right) \end{aligned}\tag{52}$$

show that the quiver algebra $\mathrm{Ext}^{\bullet}_{N^{\vee}}(\mathbb{P}^1)$ for (49) is in fact the same as $\mathrm{Ext}^{\bullet}_{PGL(2,\mathbb{C})}(G^{\vee}\times_{B^{\vee}}G^{\vee}/N^{\vee})$.

6 Complex Groups

There is a version of the conjecture for category $\mathcal{O}$, which was known to be true prior to the formulation of the conjecture for Harish-Chandra modules. Recall that for semi-simple $\mathfrak{g}$, a Borel subalgebra $\mathfrak{b}$, and a Cartan $\mathfrak{h}$, $\mathcal{O}=\mathcal{O}(\mathfrak{g},\mathfrak{b},\mathfrak{h})$ is the category with objects consisting of $\mathfrak{g}$-modules M such that M is finitely generated over $\mathfrak{g}$, locally finite over $\mathfrak{b}$, and semi-simple over $\mathfrak{h}$. We know that inside $\mathcal{O}$ there is the subcategory $\mathcal{O}_{\text{int}}$ containing modules M with highest weight space M_λ nonzero if and only if λ is integral. Then, $\mathcal{O}_{\text{int}}$ can be further decomposed:

$$\mathcal{O}_{\text{int}}=\bigoplus_{\chi\in Char_{\text{int}}Z}\mathcal{O}_\chi\supset\mathcal{O}_{\text{triv}}\tag{53}$$

Irreducible representations in $\mathcal{O}_{\text{triv}}$ are of the form $L(x\cdot 0)$, that is, they are simple modules with highest weight $x\cdot 0$ for some $x\in\mathcal{W}$, and $x\cdot\lambda=x(\lambda+\rho)-\rho$. We can further identify the irreducible $L(x\cdot 0)$ as the unique simple quotient of the corresponding Verma module

$$\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{b})}\mathbb{C}_{x\cdot 0}=\Delta(x\cdot 0)\to L(x\cdot 0).\tag{54}$$

There exists an indecomposable projective cover $P(x\cdot 0)\to\Delta(x\cdot 0)$. An old result on $\mathcal{O}$ which is related to the conjecture is the following:

Theorem 1 (Self-duality of $\mathcal{O}$) *As finite-dimensional complex algebras,*

$$\mathrm{Ext}_{\mathfrak{g}}\left(\bigoplus_{x\in\mathcal{W}}P(x\cdot 0)\right)\simeq\mathrm{Ext}^{\bullet}_{\mathcal{O}}\left(\bigoplus_{x\in\mathcal{W}}L(w_0x\cdot 0)\right),\tag{55}$$

where $w_0\in\mathcal{W}$ is the longest element of the Weyl group.

Call this algebra A. The left-hand side is not graded a priori, so this isomorphism forgets the grading. This isomorphism was conjectured by Beilinson and Ginzburg to explain inversion formulas for Kazhdan–Lusztig polynomials categorically. Soergel's conjecture is the analogous categorification of Vogan's inversion formulas for Harish-Chandra modules. The inversion formula for Kazhdan–Lusztig polynomials states that the inverse of the matrix of Kazhdan–Lusztig polynomials comes from applying the transpose and then reordering lines by w_0.

Proof (Sketch)

Step 1. Look at $P(w_0 \cdot 0)$, the indecomposable projective corresponding to the weight -2ρ, aka the "anti-dominant projective." We are interested in understanding $\mathrm{Ext}_{\mathfrak{g}}(P(w_0 \cdot 0))$. The center Z of the universal enveloping algebra surjects onto $\mathrm{Ext}_{\mathfrak{g}}(P(w_0 \cdot 0)$. On the other hand, we can embed $Z \hookrightarrow S(\mathfrak{h})$ via the Harish-Chandra morphism. We want to take the noncanonical embedding without the shift, actually, so if $S^{\mathcal{W}\cdot}$ denotes the invariants of $S(\mathfrak{h})$ with respect to the $\cdot$ action of $\mathcal{W}$, then

$$\begin{array}{ccc} Z & \longrightarrow & S(\mathfrak{h}) \\ & \overset{\simeq}{\searrow} & \uparrow \\ & & S^{\mathcal{W}\cdot} \end{array} \tag{56}$$

Then, this embedding gives us more information about the surjection

$$\begin{array}{ccc} Z & \xrightarrow{(1)} & \mathrm{Ext}_{\mathfrak{g}}(P(w_0 \cdot 0)) \\ \downarrow & \overset{(2)}{\searrow} & (3) \downarrow \simeq \\ S(\mathfrak{h}) & \longrightarrow & S/(S^+)^{\mathcal{W}} S. \end{array} \tag{57}$$

The ring $C = S/(S^+)^{\mathcal{W}} S$ is the ring of coinvariants of S.

Step 2. Define

$$\mathbb{V} = \mathrm{Hom}\big(P(w_0 \cdot 0), -\big) : \mathcal{O}_{\mathrm{triv}} \to C\text{-}mod.$$

The functor $\mathbb{V}$ is fully faithful on projectives. This tells us that the endomorphism ring has the following isomorphisms:

$$\begin{array}{ccc} \mathrm{Ext}_{\mathfrak{g}}(\bigoplus_{x \in \mathcal{W}} P(x \cdot 0)) & \xrightarrow{\simeq} & \mathrm{Ext}^{\bullet}_{\mathcal{O}}(\bigoplus_{x \in \mathcal{W}} L(w_0 x \cdot 0)) \\ \simeq \downarrow & & \\ \mathrm{Ext}_C(\bigoplus_{x \in \mathcal{W}} \mathbb{V} P(x \cdot 0)). & & \end{array} \tag{58}$$

Step 3. Now we need localization. Localization takes $\mathcal{O}_{\text{triv}}$ to $\mathscr{D}_{G/B}\text{-}mod_c$, the category of $\mathscr{D}_{G/B}$-modules which are constant on Bruhat cells. To get the true/genuine $\mathscr{D}$-modules, we really want only modules where Z acts diagonally, so take the variant $\mathcal{O}'_{\text{triv}}$ of $\mathcal{O}_{\text{triv}}$ where the Cartan is allowed to act nonsemisimply but still in a locally finite way, whereas Z is asked to strictly act by the character. These two variants of $\mathcal{O}$ happen to be equivalent. Then, taking global sections is an equivalence of categories

$$\Gamma : \mathscr{D}_{G/B}\text{-}mod \tilde{\to} \big(\mathcal{U}(\mathfrak{g})/Z^+\big)\text{-}mod, \tag{59}$$

where $Z^+ = Ann_Z(\mathbb{C}_{\text{triv}})$. The localization procedure actually goes over to derived categories:

$$\begin{array}{ccc} D^b(\mathcal{O}'_{\text{triv}}) & \xrightarrow{\simeq} & D^b(\mathscr{D}_{G/B} - mod_c) \\ & & {\scriptstyle RH}\big\downarrow {\scriptstyle \simeq} \\ & & D^{bc}_{B_u}(G/B) \end{array} \tag{60}$$

Here, B_u denotes the unipotent radical of B.
Under these equivalences, $L(w_0 x \cdot 0) \mapsto IC(\overline{BxB}/B)$. Also, at the level of derived categories, we can compute hypercohomology

$$\mathbb{H}^\bullet : D^{bc}_{B_u}(G/B) \to \mathrm{H}^\bullet(G/B)\text{-}mod,$$

taking values in graded $\mathrm{H}^\bullet(G/B)$-modules. The graded ring $\mathrm{H}^\bullet(G/B)$ is the coinvariant algebra of the dual group $C^\vee$. Also, $\mathbb{H}^\bullet$ is fully faithful on $IC(\overline{BxB}/B) =: IC_x$, which implies

$$\mathrm{Ext}^\bullet_{\mathcal{O}}\Big(\bigoplus_{x\in\mathcal{W}} L(w_0 x \cdot 0)\Big) \simeq \mathrm{Ext}_{C^\vee}\Big(\bigoplus \mathrm{H}^\bullet(IC_x)\Big). \tag{61}$$

Lastly, $\mathbb{V}P(x\cdot 0) \simeq \mathbb{H}^\bullet IC_x$. □

The relationship between this self-duality of $\mathcal{O}$ and the conjecture is seen by the equivalence

$$\mathrm{Hom}\big(\oplus P(x\cdot 0), -\big) : \mathcal{O}_{\text{triv}} \tilde{\to} A^{op}\text{-}mod^{fd}, \tag{62}$$

which occurs precisely because $\oplus P(x\cdot 0)$ is the projective generator of $\mathcal{O}_{\text{triv}}$. The intermediate step on the right-hand side was finding the equivalence to $\mathrm{Ext}^\bullet_{B_u}(G/B)$. Therefore, the theorem implies $A^{op}\text{-}mod^{fd} \simeq \mathrm{Ext}^\bullet_{B_u}(G/B)\text{-}mod^{fd}$. We know $\mathcal{O}_{\text{triv}} \subset \tilde{\mathcal{O}}_{\text{triv}}$, which was shown by Soergel to be equivalent to $\mathcal{M}(G(\mathbb{C}))_{Z^+\times Z^+\text{-triv}}$, the category of Harish-Chandra modules for some complex reductive group thought of as a real group. Then,

$$\mathcal{M}\big(G(\mathbb{C})\big)_{Z^+\times Z^+\text{-triv}} \simeq \mathrm{Ext}^\bullet_B(G/B)\text{-}\mathcal{N}il. \tag{63}$$

In general, take $G \times G$, γ acting as the split inner form and switching factors, so γ acts on $G^\vee \times G^\vee$ only by switching factors. One can check in this case that $Z^1(\Gamma; G^\vee) = G^\vee$ and therefore

$$X = \left(G^\vee \times G^\vee\right) \times_{B^\vee \times B^\vee} G^\vee = B^\vee \backslash G^\vee / B^\vee.$$

This implies that on the right-hand side of the conjecture, we have $\mathrm{Ext}^\bullet_{B^\vee \times B^\vee}(G^\vee)$-$\mathcal{N}il$, and it is easy to see that this is isomorphic to $\mathrm{Ext}^\bullet_B(G/B)$-$\mathcal{N}il$.

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Generalized Harish-Chandra Modules

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Abstract This course is an introduction to algebraic methods in the infinite-dimensional representation theory of semisimple Lie algebras over the complex numbers. In the first section we present basic definitions and theorems concerning Harish-Chandra modules, Fernando–Kac subalgebras associated to $\mathfrak{g}$-modules, generalized Harish-Chandra modules, and the special case of weight modules. Work of Kostant allows us to demonstrate that not all simple $\mathfrak{g}$-modules are generalized Harish-Chandra modules. In the second section we discuss the Zuckerman derived functors and several of their important properties. We tailor this section to the theory of algebraic constructions of generalized Harish-Chandra modules. In the third section we summarize the main results in our joint work with Ivan Penkov on the classification of generalized Harish-Chandra modules having a "generic" minimal $\mathfrak{k}$-type. This classification makes extensive use of the Zuckerman derived functors in the context of pairs $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ which is reductive in $\mathfrak{g}$. We also utilize the theory of the cohomology of the nilpotent radical of a parabolic subalgebra with coefficients in an infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module. The crucial point of this section is that we do not assume that $\mathfrak{k}$ is a symmetric subalgebra of $\mathfrak{g}$.

Keywords Cohomological induction · Zuckerman functor · Tensor products · Minimal $\mathfrak{k}$-type

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1 An Introduction to Generalized Harish-Chandra Modules

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $U\mathfrak{g}$ its universal enveloping algebra. Finite-dimensional representations of $\mathfrak{g}$ are well understood by now, but what can we say about infinite-dimensional modules? It is generally agreed that the theory of infinite-dimensional modules has revealed "wild" classification problems. That is, there is no systematic way of listing canonical forms of infinite-dimensional modules over a Lie algebra, even if they are simple, except for $\mathfrak{g} \simeq \mathfrak{sl}(2)$ (for this case,

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see [B]). A standard text on the subject is J. Dixmier's book [D]. In general, we need to focus the problem further than just the study of simple modules in order to make any progress in understanding the theory of representations of semisimple Lie algebras.

One such focus is the theory of Harish-Chandra modules ([D, Chap. 9], [Wa]).

Suppose that σ is a nontrivial automorphism of order 2 of $\mathfrak{g}$. For example, if $\mathfrak{g} = \mathfrak{sl}(n)$, let $\sigma(T) = -T^t$ for all $T \in \mathfrak{g}$. Then $\mathfrak{g}^\sigma = \mathfrak{so}(n)$. In general, we write $\mathfrak{k} = \mathfrak{g}^\sigma$ and call $\mathfrak{k}$ a *symmetric subalgebra* of $\mathfrak{g}$. The pair $(\mathfrak{g}, \mathfrak{k})$ is called a *symmetric pair*. Any symmetric subalgebra $\mathfrak{k}$ is necessarily reductive in $\mathfrak{g}$, i.e., the adjoint representation of any symmetric subalgebra $\mathfrak{k} \subset \mathfrak{g}$ on $\mathfrak{g}$ is semisimple. For a given $\mathfrak{g}$, there are finitely many conjugacy classes of symmetric subalgebras, and there is always at least one. E. Cartan classified all symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ [H, Kn1].

Definition 1.1 A Harish-Chandra module for the pair $(\mathfrak{g}, \mathfrak{k})$ is a $\mathfrak{g}$-module M satisfying:

(1) M is finitely generated over $\mathfrak{g}$.
(2) For all $v \in M$, $(U\mathfrak{k})v$ is a finite-dimensional semisimple $\mathfrak{k}$-module.
(3) For any simple, finite-dimensional $\mathfrak{k}$-module V, $\dim \mathrm{Hom}_{\mathfrak{k}}(V, M) < \infty$.

Lemma 1.2 *If M is a Harish-Chandra module for $(\mathfrak{g}, \mathfrak{k})$, then we have a canonical decomposition of the restriction of M to $\mathfrak{k}$:*

$$M \cong \bigoplus_{V \in \mathrm{Rep}\,\mathfrak{k}} \mathrm{Hom}_{\mathfrak{k}}(V, M) \otimes_{\mathbb{C}} V,$$

where $\mathrm{Rep}\,\mathfrak{k}$ *is a complete set of representatives for the isomorphism classes of simple finite-dimensional* $\mathfrak{k}$*-modules.*

See Proposition 1.13 below for a more general statement.

The following is a classical result of Harish-Chandra.

Theorem 1.3 ([HC]; see [D, Chap. 9]) *Suppose that $\mathfrak{k}$ is a symmetric subalgebra of $\mathfrak{g}$. Let M be a simple $\mathfrak{g}$-module which satisfies condition* (2) *of Definition* 1.1*. Then M satisfies condition* (3) *of Definition* 1.1.

Let us attempt to put Harish-Chandra's theorem into some perspective. By $\mathfrak{l}$ we denote an arbitrary subalgebra of $\mathfrak{g}$.

Definition 1.4 A $(\mathfrak{g}, \mathfrak{l})$-module is a $\mathfrak{g}$-module M such that for all $v \in M$, $(U\mathfrak{l})v$ is a finite-dimensional $\mathfrak{l}$-module (not necessarily semisimple over $\mathfrak{l}$).

For example, let $\mathfrak{l} = \mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, that is, a maximal solvable subalgebra. Let E be a finite-dimensional $\mathfrak{b}$-module, and let $M = M(E) = U\mathfrak{g} \otimes_{U\mathfrak{b}} E$ be the $\mathfrak{g}$-module algebraically induced from E. Algebraic induction is introduced in [D, Chap. 5].

When E is one dimensional, $M(E)$ is called a *Verma module* (although Harish-Chandra studied these objects long before Verma). For any finite-dimensional E, $M(E)$ is a $(\mathfrak{g}, \mathfrak{b})$-module; this follows from the following:

Lemma 1.5 *Let E be any finite-dimensional $\mathfrak{l}$-module. Then the induced module $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ is a $(\mathfrak{g}, \mathfrak{l})$-module.*

Proof Suppose that $Y \in \mathfrak{l}$, $u \in U\mathfrak{g}$, and $e \in E$. Then in $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ we have

$$Y(u \otimes e) = Yu \otimes e = [Y, u] \otimes e + uY \otimes e = [Y, u] \otimes e + u \otimes Ye.$$

Write $U_{\mathrm{ad}\mathfrak{l}}$ for $U\mathfrak{g}$ regarded as an $\mathfrak{l}$-module via the adjoint action $\mathfrak{l}$. The above equation implies that we have an $\mathfrak{l}$-module surjection

$$U_{\mathrm{ad}\mathfrak{l}} \otimes_{\mathbb{C}} E \to U\mathfrak{g} \otimes_{U\mathfrak{l}} E.$$

The Poincaré–Birkhoff–Witt (PBW) filtration of $U\mathfrak{g}$ yields a filtration of $U_{\mathrm{ad}\mathfrak{l}}$ by finite-dimensional $\mathfrak{l}$-submodules. Thus, $U_{\mathrm{ad}\mathfrak{l}} \otimes_{\mathbb{C}} E$ is locally finite as an $\mathfrak{l}$-module. By the above surjection, $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ is also locally finite as an $\mathfrak{l}$-module. □

Definition 1.6 Let M be a countable-dimensional $(\mathfrak{g}, \mathfrak{l})$-module, and let V be a finite-dimensional simple $\mathfrak{l}$-module.

(a) If W is a finite-dimensional $\mathfrak{l}$-submodule of M, let $[W : V]$ be the multiplicity of V as a Jordan–Hölder factor of W.
(b) Let $[M : V]$ be the supremum of $[W : V]$ as W runs over all finite-dimensional $\mathfrak{l}$-submodules of M. We call $[M : V]$ the multiplicity of V in M. We have $[M : V] \in \mathbb{N} \cup \{\omega\}$, where ω stands for the countable infinite cardinal.

Lemma 1.7 *If E is a finite-dimensional $\mathfrak{l}$-module and V is a simple finite-dimensional $\mathfrak{l}$-module, then*

$$[U\mathfrak{g} \otimes_{U\mathfrak{l}} E : V] = \left[S(\mathfrak{g}/\mathfrak{l}) \otimes_{\mathbb{C}} E : V\right].$$

Proof The PBW filtration of $U\mathfrak{g}$ is $\mathrm{ad}\mathfrak{l}$-stable. This filtration yields a filtration of $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ whose associated graded module is $S(\mathfrak{g}/\mathfrak{l}) \otimes_{\mathbb{C}} E$ (here and below $S(\ )$ stands for symmetric algebra). □

Definition 1.8 (a) A $(\mathfrak{g}, \mathfrak{l})$-module M has finite type over $\mathfrak{l}$ if $[M : V] < \infty$ for any simple finite-dimensional $\mathfrak{l}$-module V.

(b) A generalized Harish-Chandra module is a $\mathfrak{g}$-module which is of finite type for some $\mathfrak{l}$ in $\mathfrak{g}$, not necessarily specified in advance. (See [PZ1, PSZ].)

Example 1.9 Suppose that $\mathfrak{b}$ is a Borel subalgebra and E is a finite-dimensional $\mathfrak{b}$-module. Then $U\mathfrak{g} \otimes_{U\mathfrak{b}} E$ has finite type over $\mathfrak{b}$. Indeed, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{b}$. As an $\mathfrak{h}$-module, $S(\mathfrak{g}/\mathfrak{b})$ is semisimple with finite multiplicities. Hence, $S(\mathfrak{g}/\mathfrak{b})$ has finite multiplicities as a $\mathfrak{b}$-module. Now apply Lemma 1.7.

Proposition 1.10 *Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, and M be a simple $(\mathfrak{g}, \mathfrak{b})$-module.*

(a) *M is semisimple over $\mathfrak{h}$ with finite multiplicities.*
(b) *There exists a unique one-dimensional $\mathfrak{b}$-module E such that M is a quotient of $M(E) := U\mathfrak{g} \otimes_{U\mathfrak{b}} E$.*
(c) *$M(E)$ has a unique simple quotient.*

Proof Part (b): Let E be a one-dimensional $\mathfrak{b}$-submodule of M (E exists by Lie's theorem). The embedding of E into M yields a surjective homomorphism of $M(E)$ onto M.

Part (a): Follows from the same statement for $M(E)$.

Part (c): See [D, Chap. 7], [Ma]. □

Recall that $\mathfrak{l}$ is an arbitrary subalgebra of $\mathfrak{g}$ and let E be a finite-dimensional $\mathfrak{l}$-module. In general, $S(\mathfrak{g}/\mathfrak{l})$ has infinite multiplicities as an $\mathfrak{l}$-module. Hence, $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ has infinite multiplicities as an $\mathfrak{l}$-module. Now suppose that M is a simple $(\mathfrak{g}, \mathfrak{l})$-module. By Schur's lemma [D, Chap. 2, Sect. 6], the center $Z_{U\mathfrak{g}}$ of the enveloping algebra $U\mathfrak{g}$ will act via scalars on M. Let $\theta_M : Z_{U\mathfrak{g}} \to \mathbb{C}$ be the corresponding central character of M. Let E be a nonzero finite-dimensional $\mathfrak{l}$-submodule of M. Then, M is a quotient of the $(\mathfrak{g}, \mathfrak{l})$-module

$$P(E, \theta_M) = (U\mathfrak{g} \otimes_{U\mathfrak{l}} E) \otimes_{Z_{U\mathfrak{g}}} (Z_{U\mathfrak{g}}/\operatorname{Ker}\theta_M).$$

In general, $P(E, \theta_M)$ has infinite multiplicities as an $\mathfrak{l}$-module.

The following is a crucial fact leading to the proof of Harish-Chandra's theorem (Theorem 1.3).

Proposition 1.11 *If $\mathfrak{k}$ is a symmetric subalgebra of $\mathfrak{g}$, E is a simple finite-dimensional $\mathfrak{k}$-module, and θ is a homomorphism from $Z_{U\mathfrak{g}}$ to $\mathbb{C}$, then $P(E, \theta)$ has finite type over $\mathfrak{k}$.*

Proof See [D, Chap. 9], [W]. □

Proposition 1.11 implies Theorem 1.3 as, if M is a simple $(\mathfrak{g}, \mathfrak{k})$-module with central character θ_M and simple $\mathfrak{k}$-submodule E, then M is a quotient of $P(E, \theta_M)$.

Remark For general E and θ, $P(E, \theta)$ could vanish.

For later use, we state the following:

Lemma 1.12 *If $\mathfrak{l}$ is reductive in $\mathfrak{g}$ and M is a simple $(\mathfrak{g}, \mathfrak{l})$-module, then M is semisimple over $\mathfrak{l}$.*

Proof Let E be a simple $\mathfrak{l}$-submodule of M. We have a canonical homomorphism of $U\mathfrak{g} \otimes_{U\mathfrak{l}} E \to M$ given by $u \otimes e \mapsto ue$. Since M is simple, this homomorphism is surjective. In turn, the homomorphism $U_{\mathrm{ad}\,\mathfrak{l}} \otimes_{\mathbb{C}} E \to U\mathfrak{g} \otimes_{U\mathfrak{l}} E$, given by $u \otimes e \mapsto$

$u \otimes e \in U\mathfrak{g} \otimes_{U\mathfrak{l}} E$, is surjective. Since $\mathfrak{l}$ is reductive in $\mathfrak{g}$, $U_{\mathrm{ad}\,\mathfrak{l}}$ is a semisimple $\mathfrak{l}$-module; since the $\mathfrak{l}$-module E is simple, $U_{\mathrm{ad}\,\mathfrak{l}} \otimes_{\mathbb{C}} E$ is a semisimple $\mathfrak{l}$-module. Hence, $U\mathfrak{g} \otimes_{U\mathfrak{l}} E$ and M itself are semisimple over $\mathfrak{l}$. □

Proposition 1.13 *If $\mathfrak{l}$ is reductive in $\mathfrak{g}$ and M is a simple $(\mathfrak{g}, \mathfrak{l})$-module, then as an $\mathfrak{l}$-module, M is canonically isomorphic to*

$$\bigoplus_{V \in \mathrm{Rep}\,\mathfrak{l}} \mathrm{Hom}_{\mathfrak{l}}(V, M) \otimes V.$$

In particular, M has finite type over $\mathfrak{l}$ if and only if for every $V \in \mathrm{Rep}\,\mathfrak{l}$, $\mathrm{Hom}_{\mathfrak{l}}(V, M)$ is finite dimensional. In general, $[M : V] = \dim \mathrm{Hom}_{\mathfrak{l}}(V, M)$.

Proof See [Kn2]. □

Consider now the case $\mathfrak{l} = \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$, the maximal nilpotent subalgebra of a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. For any finite-dimensional $\mathfrak{n}$-module F, the algebraically induced module $U\mathfrak{g} \otimes_{U\mathfrak{n}} F$ is a $(\mathfrak{g}, \mathfrak{n})$-module, and by tensoring over $Z_{U\mathfrak{g}}$ with a finite-dimensional representation V of $Z_{U\mathfrak{g}}$,

$$(U\mathfrak{g} \otimes_{U\mathfrak{n}} F) \bigotimes_{Z_{U\mathfrak{g}}} V, \tag{1}$$

we again obtain a $(\mathfrak{g}, \mathfrak{n})$-module. Since $\mathfrak{n}$ is not symmetric, this does not guarantee finite multiplicities.

Definition 1.14 A one-dimensional $\mathfrak{n}$-module F is generic if for each simple root α of $\mathfrak{b}$ in $\mathfrak{g}$, $\mathfrak{g}_\alpha$ acts nontrivially on F (note that $\mathfrak{g}_\alpha \subset \mathfrak{n}$).

In fact, we have the following well-known result.

Theorem 1.15 [Ko] *If F is one-dimensional and generic, and V is one-dimensional, then $(U\mathfrak{g} \otimes_{U\mathfrak{n}} F) \otimes_{Z_{U\mathfrak{g}}} V$ is simple, but F occurs with infinite multiplicity.*

The reader is now urged to compare Theorem 1.15 to Proposition 1.11. In fact, one can show that the $\mathfrak{g}$-module $(U\mathfrak{g} \otimes_{U\mathfrak{n}} F) \otimes_{Z_{U\mathfrak{g}}} V$ is not a generalized Harish-Chandra module.

Let $\mathfrak{l} = \mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and M a simple $(\mathfrak{g}, \mathfrak{h})$-module. By Proposition 1.13, M is a direct sum of weight spaces (joint eigenspaces) of $\mathfrak{h}$. Each weight space corresponds to a linear functional on $\mathfrak{h}$, so we may write

$$M = \bigoplus_{\lambda \in \mathfrak{h}*} M(\lambda). \tag{2}$$

Although the dual space $\mathfrak{h}^*$ is uncountable, this sum is in fact supported on a countable set of weights, which we denote $\mathrm{supp}_{\mathfrak{h}}\, M$. In what follows we call any $\mathfrak{g}$-module satisfying (2) an $\mathfrak{h}$-*weight module* or simply a *weight module*.

Decomposition (2) should in principle allow us to understand weight modules better since the root decomposition of $\mathfrak{g}$ respects this decomposition of any $(\mathfrak{g}, \mathfrak{k})$-module M. That is, if α is a root of $\mathfrak{h}$ in $\mathfrak{g}$, X_α a nonzero root vector, then $X_\alpha M(\lambda) \subseteq M(\lambda+\alpha)$. However, there is no classification of simple weight modules for the pair $(\mathfrak{g}, \mathfrak{h})$ unless $\mathfrak{g} \simeq \mathfrak{sl}(2)$. The classification problem appears to be wild already for $\mathfrak{g} \simeq \mathfrak{sl}(3)$, although for $\mathfrak{sl}(2)$, it was solved in the 1940s and in fact is given as an exercise in [D].

Consider the case $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g} = \mathfrak{sl}(n)$ with $\mathfrak{k}$ the symmetric subalgebra consisting of elements of the type below:

$$\begin{pmatrix} * & \dots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \dots & * & 0 \\ 0 & \dots & 0 & * \end{pmatrix} \simeq \mathfrak{sl}(n-1) \oplus \mathbb{C} \tag{3}$$

(here $\mathfrak{h}$ is the subalgebra of traceless diagonal matrices).

We have the following known types of simple $(\mathfrak{g}, \mathfrak{k})$-modules (due to Kraljević [Kra]):

(1) Those which have infinite $\mathfrak{h}$-multiplicities.
(2) Modules which are $(\mathfrak{g}, \mathfrak{b})$-modules for some Borel containing $\mathfrak{h}$. Such modules have finite $\mathfrak{h}$-multiplicities.

Kraljević gives an explicit construction of a complete set of representatives for the isomorphism classes of simple $(\mathfrak{g}, \mathfrak{k})$-modules. The completeness of this set is implied by the Harish-Chandra subquotient theorem, see [D, Chap. 9].

For $\mathfrak{g} = \mathfrak{sl}(n)$, there exist simple $(\mathfrak{g}, \mathfrak{h})$-modules which are not $(\mathfrak{g}, \mathfrak{b})$-modules for any Borel $\mathfrak{b}$ containing $\mathfrak{h}$, but which have finite $\mathfrak{h}$-multiplicities. Britten and Lemire showed in 1982 that we can construct a module M such that $\operatorname{supp}_{\mathfrak{h}} M = \nu + \Lambda$, where Λ is the root lattice of $\mathfrak{h}$, and ν is a completely nonintegral weight of $\mathfrak{h}$. Moreover, if $\lambda \in \operatorname{supp}_{\mathfrak{h}} M$, then $\dim M(\lambda) = 1$. (See [BL, BBL].)

Here is an explicit construction of Britten–Lemire modules. Let $x_1, x_2, \dots, x_n$ be coordinates for $\mathbb{C}^n$. If $\lambda \in \mathfrak{h}^*$, write $\lambda = (\lambda_1,\ \lambda_2, \dots,\ \lambda_n)$, with $\lambda_1 + \lambda_2 + \dots + \lambda_n = 0$. Let x^λ denote the formal monomial $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$, and define $\frac{\partial}{\partial x_i} x^\lambda = \lambda_i x^{\lambda - (\delta_{i1}, \delta_{i2}, \dots, \delta_{in})}$ where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. Let F be the $\mathbb{C}$ vector space with basis $\{x^\lambda | \lambda \in \mathfrak{h}^*\}$. Then F is a module over the Lie algebra $\tilde{\mathfrak{g}}$ spanned by the vector fields $x_i \frac{\partial}{\partial x_j}$. Identify $\mathfrak{g} = \mathfrak{sl}(n)$ with the subalgebra $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ of $\tilde{\mathfrak{g}}$.

Then $\mathfrak{h} = \operatorname{span}\{x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\}$, and the λ-weight space of F is precisely $F(\lambda) = \mathbb{C}x^\lambda$ for each $\lambda \in \mathfrak{h}^*$. Fix a weight $\nu \in \mathfrak{h}^*$ and let $M_\nu = (U\mathfrak{g}) \cdot x^\nu \subseteq F$. Assume that $\nu_i \notin \mathbb{Z}$ for $i = 1, 2, \dots, n$. Then M_ν is a simple $(\mathfrak{g}, \mathfrak{h})$-module with $\operatorname{supp}_{\mathfrak{h}} M_\nu = \nu + \Lambda$.

For an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$ and an arbitrary $\mathfrak{g}$-module M, define

$$\mathfrak{g}[M] = \{Y \in \mathfrak{g} | \mathbb{C}Y \text{ acts locally finitely in } M\}. \tag{4}$$

Theorem 1.16 (Fernando [F], Kac, [K])[1] *The subset* $\mathfrak{g}[M]$ *is a subalgebra of* $\mathfrak{g}$, *called the Fernando–Kac subalgebra associated to* M.

Note that even for a simple $\mathfrak{g}$-module M, we may have $\mathfrak{g}[M] = 0$: such an example for $\mathfrak{g} = \mathfrak{sl}(2)$ was found by Arnal and Pinczon [AP].

Any $\mathfrak{g}$-module M is a $(\mathfrak{g}, \mathfrak{g}[M])$-module. Moreover, M is a generalized Harish-Chandra module if and only if M is of finite type over $\mathfrak{g}[M]$. (See Definition 1.8.)

At this point, we have a theory moving in two directions. To a subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, we associate the category of $(\mathfrak{g}, \mathfrak{l})$-modules. We can also associate to $\mathfrak{l}$ the subcategory of $(\mathfrak{g}, \mathfrak{l})$-modules of finite type. On the other hand, the Fernando–Kac construction allows us to identify a subalgebra $\mathfrak{g}[M]$ of $\mathfrak{g}$ for every module M. That is, if $\mathcal{C}(\mathfrak{g})$ is the category of all $\mathfrak{g}$-modules,

$$\mathfrak{g}[-] : \mathcal{C}(\mathfrak{g}) \to Sub(\mathfrak{g})$$

is a map from the class of objects of $\mathcal{C}(\mathfrak{g})$ to the set $Sub(\mathfrak{g})$ of subalgebras of $\mathfrak{g}$.

Theorem 1.17 [PS] *Let* $\mathfrak{g}$ *be reductive. Every subalgebra between* $\mathfrak{h}$ *and* $\mathfrak{g}$, *i.e., every root subalgebra of* $\mathfrak{g}$, *arises as the Fernando–Kac subalgebra of some simple weight module of* $\mathfrak{g}$.

As we will see below for $\mathfrak{g} = \mathfrak{sl}(3)$, for certain root subalgebras $\mathfrak{l} \supset \mathfrak{h}$, the equality $\mathfrak{g}[M] = \mathfrak{l}$ for a simple $\mathfrak{g}$-module M implies that M has infinite type over $\mathfrak{h}$.

Consider now the case where $\mathfrak{g}$ is simple and $\mathfrak{k}$ is a proper symmetric subalgebra of $\mathfrak{g}$. There are two cases.

(1) The center of $\mathfrak{k}$ is trivial.

In this case, $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$. If M is a simple infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module, $\mathfrak{g}[M] = \mathfrak{k}$.

(2) The center of $\mathfrak{k}$ is nontrivial. (Example: $\mathfrak{g} = \mathfrak{sl}(n)$, $\mathfrak{k} = \mathfrak{sl}(n-1) \oplus \mathbb{C}$.)

In this case, $Z\mathfrak{k} \cong \mathbb{C}$, and $\mathfrak{k}$ is the reductive part of two opposite maximal parabolic subalgebras $\mathfrak{p}_+$ and $\mathfrak{p}_-$ in $\mathfrak{g}$. Moreover, the only subalgebras lying between $\mathfrak{k}$ and $\mathfrak{g}$ are $\mathfrak{k}, \mathfrak{p}_+, \mathfrak{p}_-$, and $\mathfrak{g}$. If M is a simple infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module, $\mathfrak{g}[M]$ can be any of the three subalgebras $\mathfrak{k}, \mathfrak{p}_+$, or $\mathfrak{p}_-$. These facts are a consequence of the theory of the Fernando–Kac subalgebra plus early work of Harish-Chandra [D, Chap. 9].

The case of $\mathfrak{l}$ simple, $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}$, $\mathfrak{k} = \mathfrak{l}$ embedded diagonally into $\mathfrak{g}$ is also interesting, since $\mathfrak{l}$ is maximal in $\mathfrak{g}$.

For $\mathfrak{g} = \mathfrak{sl}(3)$, $\mathfrak{h}$ diagonal matrices, the simple $(\mathfrak{g}, \mathfrak{h})$-modules considered so far have the following Fernando-Kac subalgebras.

(1) If M is finite dimensional, then $\mathfrak{g}[M] = \mathfrak{g}$ (i.e., every $Y \in \mathfrak{g}$ acts locally finitely on M).

[1] We thank A. Joseph for pointing out that Theorem 1.16 follows also from an earlier result of B. Kostant reproduced in [GQS].

(2) If M is a simple infinite-dimensional $(\mathfrak{g}, \mathfrak{b})$-module for a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$, then $\mathfrak{b} \subseteq \mathfrak{g}[M]$, so $\mathfrak{g}[M]$ is a parabolic subalgebra of $\mathfrak{g}$.
(3) For a Britten–Lemire module M, $\mathfrak{g}[M] = \mathfrak{h}$.
(4) Let $\mathfrak{k} = \mathfrak{sl}(2) \oplus \mathbb{C}$, i.e.,

$$\mathfrak{k} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

For a simple $(\mathfrak{g}, \mathfrak{k})$-module M which is not a $(\mathfrak{g}, \mathfrak{b})$-module for any Borel subalgebra $\mathfrak{b}$, we have $\mathfrak{g}[M] = \mathfrak{k}$.

These four classes are distinguished by $\mathfrak{g}[M]$. There are many additional modules, and it is interesting to ask which subalgebras of $\mathfrak{g}$ can occur as $\mathfrak{g}[M]$.

(5) Let

$$\mathfrak{l} = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

By [F], any simple M with $\mathfrak{g}[M] = \mathfrak{l}$ has finite type over $\mathfrak{l}$.
(6) Let

$$\mathfrak{l} = \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

By [F], any simple M with $\mathfrak{g}[M] = \mathfrak{l}$ has infinite type over $\mathfrak{l}$.

Cases 1–6 exhaust all root subalgebras of $\mathfrak{sl}(3)$ up to conjugacy. Case 3, i.e., where $\mathfrak{l} = \mathfrak{h}$, is the only case where simple $(\mathfrak{g}, \mathfrak{l})$-modules M with $\mathfrak{g}[M] = \mathfrak{h}$ can have both finite or infinite type. In particular, there exist simple $\mathfrak{g}$-modules M with $\mathfrak{g}[M] = \mathfrak{h}$ which have infinite type over $\mathfrak{h}$. See [Fu1, Fu2, Fu3, PS].

The following theorem gives a general characterization of the subalgebra $\mathfrak{g}[M]$ corresponding to a generalized Harish-Chandra module M. By $\supset\!\!\!\!+$ we denote the semidirect sum of Lie algebras: the round part of the sign points toward the ideal.

Theorem 1.18 [PSZ] *Assume that $\mathfrak{g}$ is semisimple. Suppose that M is a simple generalized Harish-Chandra module.*

(a) *$\mathfrak{g}[M]$ is self-normalizing, hence $\mathfrak{g}[M]$ is algebraic. That is, $\mathfrak{g}[M]$ is the Lie algebra of a subgroup of $G = Aut(\mathfrak{g})^\circ$ ($()^\circ$ indicates the connected component of the identity).*
(b) *Let $\mathfrak{k} \subseteq \mathfrak{g}[M]$ be maximal among subalgebras of $\mathfrak{g}[M]$ that are reductive as subalgebras of $\mathfrak{g}$. Let $\mathfrak{g}[M]_{\mathrm{nil}}$ be the maximum ad-nilpotent ideal in $\mathfrak{g}[M]$. Then, $\mathfrak{g}[M] = \mathfrak{k} \supset\!\!\!\!+ \mathfrak{g}[M]_{\mathrm{nil}}$. Although $\mathfrak{k}$ is only unique up to conjugation, write $\mathfrak{k} = \mathfrak{g}[M]_{\mathrm{red}}$.*
(c) *M has finite type over $\mathfrak{g}[M]$. Moreover, $\mathfrak{g}[M]_{\mathrm{red}}$ acts semisimply on M.*
(d) *The center $Z(\mathfrak{g}[M]_{\mathrm{red}})$ of $\mathfrak{g}[M]_{\mathrm{red}}$ is equal to the centralizer $C_{\mathfrak{g}}(\mathfrak{g}[M]_{\mathrm{red}})$ of $\mathfrak{g}[M]_{\mathrm{red}}$ in $\mathfrak{g}$.*

If we are just interested in simple generalized Harish-Chandra modules, we can give a more transparent characterization of them. A simple $\mathfrak{g}$-module M is a generalized Harish-Chandra module if M is a direct sum with finite multiplicities of finite-dimensional simple $\mathfrak{g}[M]_{\text{red}}$-modules.

For M as in Theorem 1.18, there is a canonical isomorphism

$$M \xleftarrow{\sim} \bigoplus_{F \in \text{Rep}(\mathfrak{g}[M]_{\text{red}})} \text{Hom}_{\mathfrak{g}[M]_{\text{red}}}(F, M) \otimes_{\mathbb{C}} F. \tag{5}$$

Choose $\mathfrak{t}$ a Cartan subalgebra in $\mathfrak{k} = \mathfrak{g}[M]_{\text{red}}$ and fix a root order for $\mathfrak{t}$ in $\mathfrak{k}$. Then, $\text{Rep}\, \mathfrak{k}$ bijects to the dominant integral weights $\Lambda_d \subset \mathfrak{t}^*$. Therefore, we can rewrite M as

$$M \xleftarrow{\sim} \bigoplus_{\lambda \in \Lambda_d} \text{Hom}_{\mathfrak{k}}(F_\lambda, M) \otimes_{\mathbb{C}} F_\lambda \tag{6}$$

with F_λ the simple finite-dimensional module of highest weight λ. Let $M[\lambda]$ be the image of $\text{Hom}_{\mathfrak{k}}(F_\lambda, M) \otimes_{\mathbb{C}} F_\lambda$ under the isomorphism in (6). We call $M[\lambda]$ the *F_λ-isotypic $\mathfrak{k}$-submodule of M.*

Fix a subalgebra $\mathfrak{k}$ reductive in $\mathfrak{g}$. Assume (as in Theorem 1.18(d)) that $C_{\mathfrak{g}}(\mathfrak{k}) = Z(\mathfrak{k})$. How might one construct simple generalized Harish-Chandra modules M such that $\mathfrak{k} = \mathfrak{g}[M]_{\text{red}}$?

Theorem 1.19 [PSZ] *At least one simple M exists satisfying the above.*

The proof involves quite a bit of algebraic geometry. Basically, we can employ the natural action of K on certain partial flag varieties for G (here G and K are the connected algebraic groups corresponding to $\mathfrak{g}$ and $\mathfrak{k}$), then view M as global sections of a sheaf on this variety.

2 An Introduction to the Zuckerman Functor

In the following construction of $(\mathfrak{g}, \mathfrak{k})$-modules via derived functors, we will have three primary goals:

(1) Systematically construct generalized Harish-Chandra modules for this pair.
(2) Give a classification of a natural class of simple generalized Harish-Chandra modules.
(3) Calculate $\mathfrak{g}[M]$ for a class of modules M constructed via derived functors.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, with $\mathfrak{m} \subsetneq \mathfrak{k}$ a pair of subalgebras reductive in $\mathfrak{g}$. We have the category $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$ of $\mathfrak{g}$-modules which are locally finite and completely reducible over $\mathfrak{k}$, and likewise the category $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$. Both categories are closed under taking submodules, quotients, arbitrary direct sums, tensor products over $\mathbb{C}$, and $\Gamma' \,\text{Hom}_{\mathbb{C}}(-, -)$. (See the discussion below for the definition of Γ'.)

Note that $A \in \mathcal{C}(\mathfrak{g}, \mathfrak{k})$ implies $A \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$. Therefore, we can define a forgetful functor $For : \mathcal{C}(\mathfrak{g}, \mathfrak{k}) \to \mathcal{C}(\mathfrak{g}, \mathfrak{m})$. For general homological algebra reasons, this functor has a right adjoint $\Gamma : \mathcal{C}(\mathfrak{g}, \mathfrak{m}) \to \mathcal{C}(\mathfrak{g}, \mathfrak{k})$ unique up to isomorphism. Recall that Γ is a right adjoint to *For* means that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}(\mathfrak{g},\mathfrak{m})}\big(For(A), V\big) \simeq \mathrm{Hom}_{\mathcal{C}(\mathfrak{g},\mathfrak{k})}(A, \Gamma V). \tag{7}$$

An explicit construction for Γ is as follows. For $V \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$, let ΓV be the sum in V of all cyclic $\mathfrak{g}$-submodules B such that $B \in \mathcal{C}(\mathfrak{g}, \mathfrak{k})$.

Proposition 2.1 *As a set, ΓV is the set of $v \in V$ such that $(U\mathfrak{k})v$ is finite dimensional and semisimple as a $\mathfrak{k}$-module.*

Proof Suppose that, for some $v \in V$, $E = (U\mathfrak{k})v$ is finite dimensional and semisimple over $\mathfrak{k}$. Then the cyclic submodule $B = (U\mathfrak{g})v$ is a quotient of $U\mathfrak{g} \otimes_{U\mathfrak{k}} E$, which is an object in $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$. Hence, $B \in \mathcal{C}(\mathfrak{g}, \mathfrak{k})$.

Conversely, if $B = (U\mathfrak{g})v$ is in $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$, then $(U\mathfrak{k})v$ is finite dimensional and semisimple over $\mathfrak{k}$. If $v_1, v_2, \ldots, v_n$ are elements of V such that for each i, $B_i = (U\mathfrak{g})v_i$ is in $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$, then for any element w in $B_1 + B_2 + \cdots + B_n$, $(U\mathfrak{k})w$ is finite dimensional and semisimple over $\mathfrak{k}$. □

A more conceptual way to think of ΓV is as the largest $(\mathfrak{g}, \mathfrak{k})$-module in V.

The functor Γ is left exact but happens to be not right exact. To see a simple example of failure of right exactness, consider the exact sequence

$$0 \to (U\mathfrak{g})\mathfrak{g} \to U\mathfrak{g} \to \mathbb{C} \to 0. \tag{8}$$

Set $\mathfrak{k} = \mathfrak{g}$, $\mathfrak{m} = 0$. Then, Γ takes $U\mathfrak{g}$ and the augmentation ideal $(U\mathfrak{g})\mathfrak{g}$ to 0, but $\Gamma\mathbb{C} = \mathbb{C}$.

Once we know that Γ is not exact, we should have a Pavlovian response and try to define corresponding derived functors. The existence of these functors is contingent on the existence of enough injectives in the category $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$.

Lemma 2.2 *$\mathcal{C}(\mathfrak{g}, \mathfrak{m})$ has enough injectives.*

Proof Take any $V \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$. Embed V in an injective $\mathfrak{g}$-module X. Write $\Gamma' : \mathcal{C}(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g}, \mathfrak{m})$ for the adjoint of the inclusion $\mathcal{C}(\mathfrak{g}, \mathfrak{m}) \subset \mathcal{C}(\mathfrak{g})$. Here, $\Gamma' V = V$, and so $V \to \Gamma' X$ is an embedding of V to an injective object in $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$. That $\Gamma' X$ is injective is another standard fact of homological algebra which follows from Γ' being right adjoint to *For*. By repeating these steps, we can prolong the embedding $V \to \Gamma' X$ to an injective resolution

$$0 \to V \to I^0 \to I^1 \to I^2 \to \cdots \tag{9}$$

in $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$. □

Let $I^\bullet$ denote the complex $0 \to I^0 \to I^1 \to \cdots$. Now, we can define

$$R^p\Gamma(V) := \mathrm{H}^p(\Gamma I^\bullet).$$

The maps in $0 \to \Gamma I^0 \to \Gamma I^1 \to \cdots$ are maps of $(\mathfrak{g}, \mathfrak{k})$-modules; therefore $\mathrm{H}^*(\Gamma I^\bullet)$ is a $\mathbb{Z}$-graded $(\mathfrak{g}, \mathfrak{k})$-module.

We may introduce more systematic notation to identify the categories in which we are working:

$$\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{m}} : \mathcal{C}(\mathfrak{g}, \mathfrak{m}) \to \mathcal{C}(\mathfrak{g}, \mathfrak{k}). \tag{10}$$

Likewise we can write the derived functors $R\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{m}}$. In the literature, $R^*\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{m}}$ have been referred to as the Zuckerman functors, see [V]. These derived functors depend on the choice of injective resolution, but they are well defined up to isomorphism.

In $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$ we can construct a functorial resolution, by using the relative Koszul complex. Recall that $K_\bullet(\mathfrak{g}, \mathfrak{m})$ is given by $K_i(\mathfrak{g}, \mathfrak{m}) = U\mathfrak{g} \otimes_{U\mathfrak{m}} \Lambda^i(\mathfrak{g}/\mathfrak{m})$ with Koszul differential $\partial_i : K_i(\mathfrak{g}, \mathfrak{m}) \to K_{i-1}(\mathfrak{g}, \mathfrak{m})$, see [BW]. The complex $K_\bullet(\mathfrak{g}, \mathfrak{m})$ is acyclic and yields a resolution of $\mathbb{C}$. If V is a $(\mathfrak{g}, \mathfrak{m})$-module, let $I^i(V) = \Gamma' \operatorname{Hom}_{\mathbb{C}}(K_i(\mathfrak{g}, \mathfrak{m}), V)$. Then for every i, $I^i(V)$ is an injective object in $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$. Moreover, $I^\bullet(V)$ is a resolution of V. Finally, $V \rightsquigarrow I^\bullet(V)$ is an exact functor.

As an application, we can write

$$R^*\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{m}} V \cong H^*\big(\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{m}} \operatorname{Hom}_{\mathbb{C}}\big(K_\bullet(\mathfrak{g}, \mathfrak{m}), V\big)\big).$$

This formula makes clear the dependence of $R^*\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{m}}$ on the triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{m})$. Note that we have nowhere used the assumption that $\mathfrak{g}$ is finite dimensional. However, we have used in an essential way that $\mathfrak{m}$ and $\mathfrak{k}$ are finite dimensional and act semisimply on $\mathfrak{g}$ via the adjoint representation. See [PZ4] for an application to infinite-dimensional $\mathfrak{g}$.

Example 2.3 Let $\mathfrak{g}$ be semisimple, $\mathfrak{m} = \mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{k}$ be any subalgebra reductive in $\mathfrak{g}$ and containing $\mathfrak{h}$. Let $V \in \mathcal{C}(\mathfrak{g}, \mathfrak{h})$. The module V is a weight module. If V has finite $\mathfrak{h}$-multiplicities, then $R^*\Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{h}} V$ has finite $\mathfrak{k}$-multiplicities. (See Theorem 2.4 below.)

For a general triple $\mathfrak{m} \subset \mathfrak{k} \subset \mathfrak{g}$, let $\mathcal{A}(\mathfrak{g}, \mathfrak{m})$ denote the category of $(\mathfrak{g}, \mathfrak{m})$-modules which are semisimple over $\mathfrak{m}$ and have finite multiplicities.

Theorem 2.4 *Set $\Gamma = \Gamma_{\mathfrak{g},\mathfrak{k}}^{\mathfrak{g},\mathfrak{m}}$ and let $M \in \mathcal{A}(\mathfrak{g}, \mathfrak{m})$.*

(a) *For all i, $R^i\Gamma M \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$.*
(b) *If $i > \dim \mathfrak{k}/\mathfrak{m}$, then $R^i\Gamma M = 0$.*
(c) $\bigoplus_{i\in\mathbb{N}} R^i\Gamma M \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$.

Before we give the proof of Theorem 2.4, we introduce a generalization of our setup. The following more general assumptions are in effect up to Corollary 2.8 included. Assume that $\mathfrak{g}$ is finite dimensional, but no longer assume that $\mathfrak{k}$ is reductive in $\mathfrak{g}$. Let $\mathfrak{k}_r \subset \mathfrak{k}$ be maximal among subalgebras of $\mathfrak{k}$ that are reductive in $\mathfrak{g}$, and let $\mathfrak{m}$ be reductive in $\mathfrak{k}_r$. Denote by $\mathcal{C}(\mathfrak{g}, \mathfrak{k}, \mathfrak{k}_r)$ the full subcategory of $\mathfrak{g}$-modules M such that M is a $(\mathfrak{g}, \mathfrak{k})$-module which is semisimple over $\mathfrak{k}_r$.

Example 2.5 Let $\mathfrak{k} = \mathfrak{b}$, a Borel subalgebra of $\mathfrak{g}$. Choose $\mathfrak{k}_r = \mathfrak{h}$, a Cartan subalgebra of $\mathfrak{g}$ that lies in $\mathfrak{b}$. The subcategory $\mathcal{O}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h})$ of finitely generated modules in $\mathcal{C}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h})$ was introduced by Bernstein–Gelfand–Gelfand [BGG].

As before, $\mathcal{C}(\mathfrak{g}, \mathfrak{k}, \mathfrak{k}_r)$ is a full subcategory of $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$, and we can construct a right adjoint Γ to this inclusion of categories. Likewise, we can study the right derived functors of Γ. It is interesting to understand when $R^*\Gamma M$ has finite type over $\mathfrak{k}$. A simpler question is the following.

Problem Suppose that V is a simple finite-dimensional $\mathfrak{k}_r$-module. Under what conditions on V and the data above will $\dim \mathrm{Hom}_{\mathfrak{k}_r}(V, R^*\Gamma M)$ be finite?

The question of when $R^*\Gamma M$ has finite type over $\mathfrak{k}$ is related to the question of when $\dim \mathrm{Hom}_{\mathfrak{k}}(Z, R^*\Gamma M)$ is finite for finite-dimensional (not necessarily simple) $\mathfrak{k}$ modules Z.

We now establish some general properties of the functors $R^*\Gamma$.

Proposition 2.6 *Suppose that $M \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$ and W is a finite-dimensional $\mathfrak{g}$-module. Then for every $i \in \mathbb{N}$, we have a natural isomorphism $W \otimes_{\mathbb{C}} R^i\Gamma M \cong R^i\Gamma(W \otimes_{\mathbb{C}} M)$.*

Proof First we prove that if $N \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$, we have a natural isomorphism $W \otimes_{\mathbb{C}} \Gamma N \cong \Gamma(W \otimes_{\mathbb{C}} N)$: Since W is finite dimensional and ΓN is locally finite over $\mathfrak{k}$, we have a natural injective map from $W \otimes_{\mathbb{C}} \Gamma N$ into $\Gamma(W \otimes_{\mathbb{C}} N)$. Suppose that Z is any finite-dimensional $\mathfrak{k}$-module. $\mathrm{Hom}_{\mathfrak{k}}(Z, -)$ is a left exact functor. Hence, we obtain a natural injective map

$$\alpha_Z : \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} \Gamma N) \to \mathrm{Hom}_{\mathfrak{k}}\big(Z, \Gamma(W \otimes_{\mathbb{C}} N)\big).$$

Now,

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} \Gamma N) &\cong \mathrm{Hom}_{\mathfrak{k}}(Z \otimes_{\mathbb{C}} W^*, \Gamma N)\\ &\cong \mathrm{Hom}_{\mathfrak{k}}(Z \otimes_{\mathbb{C}} W^*, N)\\ &\cong \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} N).\end{aligned}$$

Meanwhile, $\mathrm{Hom}_{\mathfrak{k}}(Z, \Gamma(W \otimes_{\mathbb{C}} N)) \cong \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} N)$. Hence, we have a commutative diagram, with vertical isomorphisms,

$$\begin{array}{ccc}\mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} \Gamma N) & \overset{\alpha_Z}{\hookrightarrow} & \mathrm{Hom}_{\mathfrak{k}}\big(Z, \Gamma(W \otimes_{\mathbb{C}} N)\big)\\ \downarrow s & & \downarrow s\\ \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} N) & \overset{\beta_Z}{\to} & \mathrm{Hom}_{\mathfrak{k}}(Z, W \otimes_{\mathbb{C}} N),\end{array}$$

where β_Z is the map induced by α_Z.

We claim that β_Z is the identity map. This follows from the canonical construction of the diagram. Hence, α_Z is an isomorphism. Since $W \otimes_{\mathbb{C}} \Gamma N$ and $\Gamma(W \otimes_{\mathbb{C}} N)$ are both locally finite over $\mathfrak{k}$, it follows that the injection of $W \otimes_{\mathbb{C}} \Gamma N$ into $\Gamma(W \otimes_{\mathbb{C}} N)$ is an isomorphism.

Next, we choose $i \in \mathbb{N}$ and $I^\bullet$ a resolution of M by injective objects in $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$. We have $W \otimes_{\mathbb{C}} R^i \Gamma M \cong W \otimes_{\mathbb{C}} H^i(\Gamma I^\bullet) \cong H^i(W \otimes_{\mathbb{C}} \Gamma I^\bullet)$, since $W \otimes_{\mathbb{C}} (-)$ is an exact functor.

Thus, by the first part of the proof, $W \otimes_{\mathbb{C}} R^i \Gamma M \cong H^i(\Gamma(W \otimes_{\mathbb{C}} I^\bullet))$. Observe that $W \otimes_{\mathbb{C}} I^j$ is an injective object in $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$: if Q is a module in $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$, $\mathrm{Hom}_{\mathfrak{g}}(Q, W \otimes_{\mathbb{C}} I^j) \cong \mathrm{Hom}_{\mathfrak{g}}(Q \otimes_{\mathbb{C}} W^*, I^j)$; hence, $Q \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(Q, W \otimes_{\mathbb{C}} I^j)$ is an exact functor on $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$. Thus, $W \otimes_{\mathbb{C}} I^\bullet$ is a resolution of $W \otimes_{\mathbb{C}} M$ by injective objects in $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$, and $H^i(\Gamma(W \otimes_{\mathbb{C}} I^\bullet)) \cong R^i \Gamma(W \otimes_{\mathbb{C}} M)$. □

For any $\mathfrak{g}$-module N and any element $z \in Z_{U\mathfrak{g}}$, we write z_N for the $\mathfrak{g}$-module endomorphism of N defined by $z_N v = zv$ for $v \in N$.

Proposition 2.7 *Let $M \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$. Then for every $i \in \mathbb{N}$, $z_{R^i \Gamma M} = R^i \Gamma z_M$.*

Proof Let $I^\bullet$ be a resolution of M in $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$ by injective objects. The chain map $z_{I^\bullet}$ is a lifting of z_M to the resolution $I^\bullet$. The morphism $R^i \Gamma z_M$ is the action of z on $R^i \Gamma M$ induced by the chain map $z_{I^\bullet}$. Finally, the morphism $z_N : N \to N$ is natural in N. It follows that $R^i \Gamma z_M = z_{R^i \Gamma M}$. □

Now suppose that $\mathfrak{a}$ is an ideal in $Z_{U\mathfrak{g}}$; for any $\mathfrak{g}$-module N, let $N^{\mathfrak{a}} = \{v \in N | \mathfrak{a} v = 0\}$.

Corollary 2.8 *If $M \in \mathcal{C}(\mathfrak{g}, \mathfrak{m})$ and $M^{\mathfrak{a}} = M$, then $(R^* \Gamma M)^{\mathfrak{a}} = R^* \Gamma M$.*

Let us now return to the setup when $\mathfrak{k}$ is reductive in $\mathfrak{g}$.

Proof of Theorem 2.4 Let V be a finite-dimensional simple $\mathfrak{k}$-module. We will study $\mathrm{Hom}_{\mathfrak{k}}(V, R^i \Gamma M)$. Let $I^\bullet$ be a resolution of M by injective objects in $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$. By definition, $\mathrm{Hom}_{\mathfrak{k}}(V, R^i \Gamma M) \cong \mathrm{Hom}_{\mathfrak{k}}(V, H^i(\Gamma I^\bullet))$; but $\mathrm{Hom}_{\mathfrak{k}}(V, -)$ is an exact functor in the category $\mathcal{C}(\mathfrak{g}, \mathfrak{k})$, and hence $\mathrm{Hom}_{\mathfrak{k}}(V, R^i \Gamma M) \cong H^i(\mathrm{Hom}_{\mathfrak{k}}(V, \Gamma I^\bullet))$.

By Proposition 2.1, we have $\mathrm{Hom}_{\mathfrak{k}}(V, \Gamma J) \cong \mathrm{Hom}_{\mathfrak{k}}(V, J)$ for any $(\mathfrak{g}, \mathfrak{m})$-module J. Hence, $\mathrm{Hom}_{\mathfrak{k}}(V, R^i \Gamma M) \cong H^i(\mathrm{Hom}_{\mathfrak{k}}(V, I^\bullet))$.

Next, we observe that since I^j is an injective object in $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$, I^j is an injective object in $\mathcal{C}(\mathfrak{k}, \mathfrak{m})$. To see this, let N be an object in $\mathcal{C}(\mathfrak{k}, \mathfrak{m})$. Then $\mathrm{Hom}_{\mathfrak{k}}(N, I^j) \cong \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g} \otimes_{U\mathfrak{k}} N, I^j)$. Note that $U\mathfrak{g} \otimes_{U\mathfrak{k}} N$ is an object in $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$. The functor $U\mathfrak{g} \otimes_{U\mathfrak{k}} (-)$ is exact. Also, the functor $\mathrm{Hom}_{\mathfrak{g}}(-, I^j)$ from $\mathcal{C}(\mathfrak{g}, \mathfrak{m})$ to $\mathbb{C}$-mod is exact. Hence, by the above natural isomorphism, the functor $\mathrm{Hom}_{\mathfrak{k}}(-, I^j)$ is exact. Therefore I^j is an injective object in $\mathcal{C}(\mathfrak{k}, \mathfrak{m})$, and $I^\cdot$ is an injective resolution of M in $\mathcal{C}(\mathfrak{k}, \mathfrak{m})$.

We now conclude that $\mathrm{Hom}_{\mathfrak{k}}(V, R^* \Gamma M) \cong \mathrm{Ext}^*_{\mathcal{C}(\mathfrak{k}, \mathfrak{m})}(V, M)$.

If $\mathfrak{m} = 0$, classical homological algebra tells us that $\mathrm{Ext}^*_{\mathfrak{k}}(V, M) \cong H^*(\mathfrak{k}, V^* \otimes_{\mathbb{C}} M)$, where $H^*(\mathfrak{k}, -)$ is Lie algebra cohomology [BW]. Since $\mathfrak{m}$ is reductive in $\mathfrak{k}$, we have

$$\mathrm{Ext}^*_{\mathcal{C}(\mathfrak{k}, \mathfrak{m})}(V, M) \cong H^*(\mathfrak{k}, \mathfrak{m}, V^* \otimes_{\mathbb{C}} M) = H^*\big(\mathrm{Hom}_{\mathfrak{m}}\big(\Lambda^\bullet(\mathfrak{k}/\mathfrak{m}), V^* \otimes_{\mathbb{C}} M\big)\big),$$

where the complex $\mathrm{Hom}_{\mathfrak{m}}(\Lambda^{\bullet}(\mathfrak{k}/\mathfrak{m}), V^{*} \otimes_{\mathbb{C}} M)$ is endowed with the relative Koszul differential.

Part (b) of Theorem 2.4 is now immediate, since $\Lambda^{i}(\mathfrak{k}/\mathfrak{m}) = 0$ for $i > \dim(\mathfrak{k}/\mathfrak{m})$; thus, for any simple finite-dimensional $\mathfrak{k}$-module V,

$$\mathrm{Hom}_{\mathfrak{k}}\big(V, R^{i}\Gamma M\big) = 0 \quad \text{for } i > \dim(\mathfrak{k}/\mathfrak{m}).$$

By assumption, M has finite type over $\mathfrak{m}$, V is finite dimensional, and $\Lambda^{i}(\mathfrak{k}/\mathfrak{m})$ is finite dimensional. Hence,

$$\dim \mathrm{Hom}_{\mathfrak{m}}\big(\Lambda^{i}(\mathfrak{k}/\mathfrak{m}), V^{*} \otimes M\big) = \dim \mathrm{Hom}_{\mathfrak{m}}\big(V \otimes_{\mathbb{C}} \Lambda^{i}(\mathfrak{k}/\mathfrak{m}), M\big) < \infty.$$

From the isomorphisms proved above we conclude that part (a) holds. Finally, parts (a) and (b) imply part (c). □

Definition 2.9 A $\mathfrak{g}$-module N is locally $Z_{U\mathfrak{g}}$-finite if for any $v \in N$, $Z_{U\mathfrak{g}}v$ is finite dimensional.

If $\theta : Z_{U\mathfrak{g}} \to \mathbb{C}$ is a homomorphism and N is a $\mathfrak{g}$-module, we set $P_{\theta}N = \bigcup_{s\in\mathbb{N}} N^{(\mathrm{Ker}\,\theta)^{s}}$. Observe that $P_{\theta}N$ is a $\mathfrak{g}$-submodule of N. By C we denote the set of homomorphisms of $Z_{U\mathfrak{g}}$ to $\mathbb{C}$ (central characters).

Lemma 2.10 *If a $U\mathfrak{g}$-module N is locally $Z_{U\mathfrak{g}}$-finite, then $N = \bigoplus_{\theta\in C} P_{\theta}N$.*

Proof By definition, $\bigoplus_{\theta\in C} P_{\theta}N \subset N$. To show the lemma, note that for any $v \in N$, $Z_{U\mathfrak{g}}v$ is a finite-dimensional $Z_{U\mathfrak{g}}$-submodule of N. By decomposing v as a sum of generalized $Z_{U\mathfrak{g}}$-eigenvectors, we obtain $v \in \bigoplus_{\theta\in C} P_{\theta}N$. □

Proposition 2.11 *$R^{i}\Gamma$ commutes with inductive limits.*

Proof Let V be a finite-dimensional simple $\mathfrak{k}$-module. By the proof of Theorem 2.4, we have a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{k}}\big(V, R^{i}\Gamma M\big) \cong H^{i}\big(\mathrm{Hom}_{\mathfrak{m}}\big(\Lambda^{\bullet}(\mathfrak{k}/\mathfrak{m}), V^{*} \otimes_{\mathbb{C}} M\big)\big).$$

Since $\Lambda^{\bullet}(\mathfrak{k}/\mathfrak{m})$ is finite dimensional, the functor $H^{i}(\mathrm{Hom}_{\mathfrak{m}}(\Lambda^{\bullet}(\mathfrak{k}/\mathfrak{m}), V^{*} \otimes_{\mathbb{C}} (-)))$ commutes with inductive limits. Hence, the functor $\mathrm{Hom}_{\mathfrak{k}}(V, R^{i}\Gamma)$ commutes with inductive limits. Finally, $R^{i}\Gamma \cong \bigoplus_{V} \mathrm{Hom}_{\mathfrak{k}}(V, R^{i}\Gamma)$, where V runs over finite-dimensional simple $\mathfrak{k}$-modules. □

Proposition 2.12 *If $M \in C(\mathfrak{g}, \mathfrak{m})$ is locally $Z_{U\mathfrak{g}}$-finite, then so is $R^{i}\Gamma M$ for any i. Moreover, for any $\theta \in C$, $P_{\theta}(R^{i}\Gamma M) = R^{i}\Gamma(P_{\theta}M)$.*

Proof If $M \in C(\mathfrak{g}, \mathfrak{m})$ is locally $Z_{U\mathfrak{g}}$-finite, then M is an inductive limit of submodules annihilated by an ideal of finite codimension in $Z_{U\mathfrak{g}}$. By Corollary 2.8 and Proposition 2.11, $R^{i}\Gamma M$ is likewise an inductive limit of modules annihilated by an ideal of finite codimension in $Z_{U\mathfrak{g}}$. Hence, $R^{i}\Gamma M$ is locally $Z_{U\mathfrak{g}}$-finite. Moreover, Corollary 2.8 and Lemma 2.10 allow us to conclude that $P_{\theta}(R^{i}\Gamma M) = R^{i}\Gamma(P_{\theta}M)$ for any $\theta \in C$. □

Proposition 2.13 *For $M \in \mathcal{A}(\mathfrak{g}, \mathfrak{m})$ and $V \in \operatorname{Rep}\mathfrak{k}$, we have*

$$\sum_i (-1)^i \dim \operatorname{Hom}_{\mathfrak{k}}\big(V, R^i\Gamma M\big) = \sum_i (-1)^i \dim \operatorname{Hom}_{\mathfrak{m}}\big(V \otimes_{\mathbb{C}} \Lambda^i(\mathfrak{k}/\mathfrak{m}), M\big).$$

In particular, the alternating sum on the left depends only on the restriction of M to $\mathfrak{m}$.

Proof By the proof of Theorem 2.4 we have

$$\dim \operatorname{Hom}_{\mathfrak{k}}\big(V, R^i\Gamma M\big) = \dim H^i\big(\operatorname{Hom}_{\mathfrak{m}}\big(V \otimes_{\mathbb{C}} \Lambda^\bullet(\mathfrak{k}/\mathfrak{m}), M\big)\big) < \infty.$$

The proof of Proposition 2.13 now follows from the Euler–Poincaré principle. □

Remark The dimension of $\operatorname{Hom}_{\mathfrak{k}}(V, R^i\Gamma M) \cong H^i(\mathfrak{k}, \mathfrak{m}, V^* \otimes_{\mathbb{C}} M)$ will in general depend on the $\mathfrak{g}$-module M and not just on its restriction to $\mathfrak{m}$.

Remark Theorem 2.4(b) holds for $M \in C(\mathfrak{g}, \mathfrak{m})$. In particular, we obtain the following.

Corollary 2.14 *If $M \in C(\mathfrak{g}, \mathfrak{m})$ and $i > \dim \mathfrak{k}/\mathfrak{m}$, then*

$$H^i\big(\Gamma \operatorname{Hom}\big(K_\bullet(\mathfrak{g}, \mathfrak{m}), M\big)\big) = 0.$$

Note that $K_i(\mathfrak{g}, \mathfrak{m}) = 0$ if and only if $i > \dim(\mathfrak{g}/\mathfrak{m})$.

Even if M is simple over $U\mathfrak{g}$, we can have $R^p\Gamma M$ reducible over $U\mathfrak{g}$. We do not have to look hard for examples.

Example 2.15 Recall that $\mathfrak{m}$ is reductive in $\mathfrak{k}$. Let $n = \dim(\mathfrak{k}/\mathfrak{m})$. Then $H^n(\mathfrak{k}, \mathfrak{m}, \mathbb{C}) \cong \mathbb{C}$. Hence, $R^n\Gamma\mathbb{C} \cong \mathbb{C}$. Thus, the inequality for the vanishing of $R^i\Gamma M$ in Theorem 2.4(b) is sharp.

Example 2.16 Assume that $\mathfrak{m} = \mathfrak{t}$, a Cartan subalgebra of $\mathfrak{k}$. Let K be a connected complex algebraic group with Lie algebra $\mathfrak{k}$, and let T be the subgroup of K with Lie algebra $\mathfrak{t}$. Let K_0 be a maximal compact subgroup of K; choose K_0 so that $K_0 \cap T = T_0$ is a maximal torus in T. We have $H^*(K_0/T_0) \cong H^*(\mathfrak{k}, \mathfrak{t}, \mathbb{C})$. If $l = \operatorname{rk}\mathfrak{k}_{ss}$, then $\dim H^2(K_0/T_0) = l$. Now, as a $\mathfrak{k}$-module, $R^*\Gamma\mathbb{C} \cong H^*(\mathfrak{k}, \mathfrak{m}, \mathbb{C})$ with the trivial action of $\mathfrak{k}$. Hence $\dim R^2\Gamma\mathbb{C} = l$, and thus $R^2\Gamma\mathbb{C}$ is in general a reducible trivial $\mathfrak{k}$-module. In fact, $R^2\Gamma\mathbb{C}$ is in general a reducible trivial $\mathfrak{g}$-module.

Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and extend it to be a Cartan subalgebra $\mathfrak{h}$ on $\mathfrak{g}$. Let M be a simple module in $\mathcal{A}(\mathfrak{g}, \mathfrak{t})$. Then M is automatically in $\mathcal{A}(\mathfrak{g}, \mathfrak{h})$. Simple modules in $\mathcal{A}(\mathfrak{g}, \mathfrak{h})$ have been classified by Fernando [F] and Mathieu [M]. It is an open problem to determine which simple $(\mathfrak{g}, \mathfrak{h})$-modules of finite type over $\mathfrak{h}$ are also of finite type over $\mathfrak{t}$. We want to study $R^*\Gamma^{\mathfrak{g},\mathfrak{t}}_{\mathfrak{g},\mathfrak{k}} M$ when M is a module in $\mathcal{A}(\mathfrak{g}, \mathfrak{t})$.

Consider first the case where $\mathfrak{h} = \mathfrak{t}$, i.e., where $\mathfrak{k}$ is a root subalgebra of $\mathfrak{g}$. As an interesting exercise, for the example $\mathfrak{g} = \mathfrak{sl}(n)$ and M a Britten–Lemire module, one can show that $R^*\Gamma^{\mathfrak{g},\mathfrak{h}}_{\mathfrak{g},\mathfrak{k}} M = 0$. If we take some Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $\mathfrak{h}$ and choose a weight $\lambda \in \mathfrak{h}^*$, we can study $R^*\Gamma^{\mathfrak{g},\mathfrak{h}}_{\mathfrak{g},\mathfrak{k}} \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{b}} \mathbb{C}_\lambda$. This is a family of graded $(\mathfrak{g}, \mathfrak{k})$-modules in $\mathcal{A}(\mathfrak{g}, \mathfrak{k})$. Let us examine the behavior of these $(\mathfrak{g}, \mathfrak{k})$-modules in some examples.

Example 2.17 Suppose that $\mathfrak{k} = \mathfrak{g}$. Then $R^*\Gamma \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{b}} \mathbb{C}_\lambda \in \mathcal{A}(\mathfrak{g}, \mathfrak{g})^*$. Either all the derived functor modules vanish, or for exactly one degree, say $i(\lambda)$, $R^{i(\lambda)}\Gamma \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{b}} \mathbb{C}_\lambda \neq 0$ and is a simple $\mathfrak{g}$-module. This is the situation considered in the Borel–Weil–Bott theorem, see [EW].

Example 2.18 Let $\mathfrak{g} = \mathfrak{sl}(n)$, $n = p + q$ with $p, q > 0$, $\mathfrak{k} = \mathfrak{s}(\mathfrak{gl}(p) \oplus \mathfrak{gl}(q)) := \{m \oplus n \mid \mathrm{tr}\, m = -\mathrm{tr}\, n\}$, and $\mathfrak{h}$ the diagonal Cartan subalgebra. Choose some Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$, and also choose $\lambda \in \mathfrak{h}^*$. Let $s = \frac{1}{2} \dim \mathfrak{k}/\mathfrak{h}$. Then,

$$A_{\mathfrak{b}}(\lambda) := R^s \Gamma\, \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{b}} \mathbb{C}_\lambda$$

can either be 0 or an infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module [VZ]. It may be simple or reducible, and if reducible, it may not be semisimple over $U\mathfrak{g}$. What are the possibilities for $\mathfrak{g}[A_{\mathfrak{b}}(\lambda)]$? There are three possibilities:

(1) $\mathfrak{g}[A_{\mathfrak{b}}(\lambda)] = \mathfrak{k}$,
(2) $\mathfrak{g}[A_{\mathfrak{b}}(\lambda)] = \mathfrak{k} \supsetplus \tilde{\mathfrak{n}}^+$,
(3) $\mathfrak{g}[A_{\mathfrak{b}}(\lambda)] = \mathfrak{k} \supsetplus \tilde{\mathfrak{n}}^-$,

where $\tilde{\mathfrak{n}}^+$ is the nilradical for a maximal parabolic containing $\mathfrak{k}$, and $\tilde{\mathfrak{n}}^-$ is its opposite. Conversely, for each of these choices, we can give a pair $(\mathfrak{b}, \lambda)$ so that $\mathfrak{g}[A_{\mathfrak{b}}(\lambda)]$ is that subalgebra. For this fact, see [PZ4].

Consider now the general case where $\mathfrak{h} \neq \mathfrak{t}$. Let $\mathfrak{g}$ be any semisimple Lie algebra and, in addition to $\mathfrak{t}$ and $\mathfrak{k}$, consider an arbitrary parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ with $\mathfrak{p} \supset \mathfrak{h}$. Let $N \in \mathcal{A}(\mathfrak{p}, \mathfrak{t})$ and consider $R^*\Gamma^{\mathfrak{g},\mathfrak{t}}_{\mathfrak{g},\mathfrak{k}} \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}} N$. Note that $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}} N$ may or may not be in $\mathcal{A}(\mathfrak{g}, \mathfrak{t})$. The parabolic subalgebra $\mathfrak{p}$ has a Levi decomposition $\mathfrak{p} = \mathfrak{p}_{\mathrm{red}} \supsetplus \mathfrak{n}_{\mathfrak{p}}$, and as a vector space we can write $\mathfrak{g} = \mathfrak{n}^-_{\mathfrak{p}} \oplus \mathfrak{p}_{\mathrm{red}} \oplus \mathfrak{n}_{\mathfrak{p}}$. Then, $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}} N \simeq U(\mathfrak{n}^-_{\mathfrak{p}}) \otimes_{\mathbb{C}} N$. In this presentation, $\mathfrak{t}$ can be seen to act by the adjoint action on $\mathfrak{n}^-_{\mathfrak{p}}$. There are choices where $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}} N$ is not in $\mathcal{A}(\mathfrak{g}, \mathfrak{t})$, but $R^*\Gamma\, \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}} N \in \mathcal{A}(\mathfrak{g}, \mathfrak{k})$.

For example, let $\mathfrak{l}$ be simple, $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}$, and $\mathfrak{k}$ the diagonal embedding of $\mathfrak{l}$ into $\mathfrak{g}$. Let $\mathfrak{t}$ be the Cartan of $\mathfrak{k}$, and $\mathfrak{p}$ the sum of a Borel of $\mathfrak{l}$ in the first factor with the opposite Borel in the second factor.

Definition 2.19 Given a triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{t})$, a $\mathfrak{t}$-compatible parabolic subalgebra is any parabolic subalgebra of the form

$$\mathfrak{p}_h = \bigoplus_{\mathrm{Re}\,\alpha(h) \geq 0} \mathfrak{g}_\alpha \tag{11}$$

for a fixed $h \in \mathfrak{t}$.

In this definition, $\mathfrak{g}_\alpha$ is the α-weight space for $\mathfrak{t}$ acting on $\mathfrak{g}$. Also, $\mathfrak{t} \subseteq \mathfrak{g}_0 \subseteq \mathfrak{p}_h$, $\mathfrak{p}_{h,\mathrm{red}} = \bigoplus_{\mathrm{Re}\,\alpha(h)=0} \mathfrak{g}_\alpha$, and $\mathfrak{n}_{\mathfrak{p}_h} = \bigoplus_{\mathrm{Re}\,\alpha(h)>0} \mathfrak{g}_\alpha$

Lemma 2.20 *Let $\mathfrak{p}$ be a $\mathfrak{t}$-compatible parabolic subalgebra. Assume that $\mathfrak{p} = \mathfrak{p}_h$ and h acts by a scalar in $N \in \mathcal{A}(\mathfrak{p}, \mathfrak{t})$. Then* $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N \in \mathcal{A}(\mathfrak{g}, \mathfrak{t})$.

Proof By the same argument as in the proof of Lemma 1.5, if N is a $(\mathfrak{p}, \mathfrak{t})$-module, then $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$ is a $(\mathfrak{g}, \mathfrak{t})$-module. Moreover, by the same argument as in the proof of Lemma 1.7, $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N \cong S(\mathfrak{g}/\mathfrak{p}) \otimes_{\mathbb{C}} N$ as a $(\mathfrak{t}, \mathfrak{t})$-module. By the assumption that $\mathfrak{p} = \mathfrak{p}_h$, the eigenspaces of h in $S(\mathfrak{g}/\mathfrak{p})$ are finite dimensional. By the assumption on N, h acts by a scalar in N. It follows that the weight spaces of $\mathfrak{t}$ in $S(\mathfrak{g}/\mathfrak{p}) \otimes_{\mathbb{C}} N$ are finite dimensional. □

Consider $R^* \Gamma \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$ for $\mathfrak{p}$ as above. This is a generalized Harish-Chandra module, but we also have a vanishing theorem which tells us that $R^i \Gamma \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N = 0$ if $i < s := \frac{1}{2} \dim \mathfrak{k}/\mathfrak{t}$. This, along with an earlier result, indicates that the only possibly nonvanishing derived functors occur for $s \le i \le 2s$. The proof of vanishing is given in [V, Chap. 6]; see also [PZ3]. This construction is known as cohomological induction, see also [KV].

Consider now the special case where $\mathfrak{k}$ is isomorphic to $\mathfrak{sl}(2)$. There are finitely many conjugacy classes of such subalgebras, classified by Dynkin [Dy]. Examples include:

- $\mathfrak{g} = \mathfrak{sl}(3)$, $\mathfrak{k} = \mathfrak{so}(3) \simeq \mathfrak{sl}(2)$; this pair is symmetric.
- $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, $\mathfrak{k} = \mathfrak{sl}(2) = \{(Y, Y) \in \mathfrak{g} | Y \in \mathfrak{sl}(2)\}$; this pair is symmetric.
- $\mathfrak{g} = \mathfrak{sp}(4)$, $\mathfrak{k} = \mathfrak{sl}(2)$, the principal $\mathfrak{sl}(2)$; this pair is not symmetric.
- $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, with $\mathfrak{k} = \mathfrak{sl}(2) = \{(Y, Y, Y) \in \mathfrak{g} | Y \in \mathfrak{sl}(2)\}$; this pair is not symmetric.

Since $\mathfrak{t}$ is one-dimensional when $\mathfrak{k} = \mathfrak{sl}(2)$, we have $s = 1$, so we need only study $i = 1, 2$. The duality theorem for relative Lie algebra cohomology [BW] implies that if $\Gamma(\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N)^* = 0$, then $R^2 \Gamma \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N = 0$. So, we need only study $R^1 \Gamma\, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$. This module may still be 0. We will try to understand $R^1 \Gamma \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$ by using an Euler characteristic trick. Given a module V for $\mathfrak{k} \simeq \mathfrak{sl}(2)$, we have

$$\dim \mathrm{Hom}_{\mathfrak{k}}\big(V, R^1 \Gamma\, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N\big) = -\sum_j (-1)^j \dim \mathrm{H}^j \big(\mathfrak{k}, \mathfrak{t}; V^* \otimes_{\mathbb{C}} \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N\big). \tag{12}$$

This formula is an immediate consequence of Proposition 2.13. We should think of (12) as analogous to the study of cohomology of surfaces with coefficients in locally constant sheaves. Using formula (12), one can in specific cases show that $R^1 \Gamma\, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$ is nonzero.

Since $\dim \mathfrak{t} = 1$, $\mathfrak{p} = \mathfrak{p}_h$ is one of two parabolic subalgebras. In either case, $(\mathfrak{p}_h)_{\mathrm{red}} \simeq C_{\mathfrak{g}}(\mathfrak{t})$, and the nilpotent part depends on the decomposition of $\mathfrak{g}$ into weights under the action of $\mathfrak{t}$. In general, $\mathfrak{p}$ is not a Borel subalgebra. Let N be a finite-dimensional module over $C_{\mathfrak{g}}(\mathfrak{t})$, with trivial action of $\mathfrak{n}_{\mathfrak{p}}$. Consider $R^1 \Gamma\, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}} N$. We can use formula (12) to compute the $\mathfrak{k}$-multiplicities as a sequence of natural numbers, see [PZ1].

3 Construction and Reconstruction of Generalized Harish-Chandra Modules

In this final section we give an introduction to the results of [PZ3]. These results provide a classification of a class of simple generalized Harish-Chandra modules.

We start by introducing a notation. A *multiset* is a function f from a set D into $\mathbb{N}$. A *submultiset* of f is a multiset f' defined on the same domain D such that $f'(d) \le f(d)$ for any $d \in D$. For any finite multiset f, defined on an additive monoid D, we can put $\rho_f := \frac{1}{2}\sum_{d\in D} f(d)d$.

We assume that the quadruple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \mathfrak{t})$ as in the previous section is fixed. We assume further that $\mathfrak{k}$ is algebraic in $\mathfrak{g}$. If $M = \bigoplus_{\omega\in\mathfrak{t}^*} M(\omega)$ is a $\mathfrak{t}$-weight module for which all $M(\omega)$ are finite dimensional, M determines the multiset $\mathrm{ch}_\mathfrak{t} M$ which is the function $\omega \mapsto \dim M(\omega)$ defined on the set of $\mathfrak{t}$-weights of M.

Note that the $\mathbb{R}$-span of the roots of $\mathfrak{h}$ in $\mathfrak{g}$ fixes a real structure on $\mathfrak{h}^*$, whose projection onto $\mathfrak{t}^*$ is a well-defined real structure on $\mathfrak{t}^*$. In what follows, we will denote by $\mathrm{Re}\,\lambda$ the real part of an element $\lambda \in \mathfrak{t}^*$. We fix also a Borel subalgebra $\mathfrak{b}_\mathfrak{k} \subset \mathfrak{k}$ with $\mathfrak{b}_\mathfrak{k} \supset \mathfrak{t}$. Then $\mathfrak{b}_\mathfrak{k} = \mathfrak{t} \supsetplus \mathfrak{n}_\mathfrak{k}$, where $\mathfrak{n}_\mathfrak{k}$ is the nilradical of $\mathfrak{b}_\mathfrak{k}$. We set $\rho := \rho_{\mathrm{ch}_\mathfrak{t}\mathfrak{n}_\mathfrak{k}}$ and $\rho_\mathfrak{n}^\perp = \rho_{\mathrm{ch}_\mathfrak{t}(\mathfrak{n}\cap\mathfrak{k}^\perp)}$.

Let $\langle\cdot,\cdot\rangle$ be the unique $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}^*$ such that $\langle\alpha,\alpha\rangle = 2$ for any long root of a simple component of $\mathfrak{g}$. The form $\langle\cdot,\cdot\rangle$ enables us to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then $\mathfrak{h}$ is identified with $\mathfrak{h}^*$, and $\mathfrak{k}$ is identified with $\mathfrak{k}^*$. The superscript $\perp$ indicates orthogonal space. Note that there is a canonical $\mathfrak{k}$-module decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$. We also set $\|\kappa\|^2 := \langle\kappa,\kappa\rangle$ for any $\kappa \in \mathfrak{h}^*$.

We say that an element $\lambda \in \mathfrak{t}^*$ is $(\mathfrak{g},\mathfrak{k})$-*regular* if $\langle\mathrm{Re}\,\lambda,\alpha\rangle \neq 0$ for nonzero $\mathfrak{t}$-weights α of $\mathfrak{g}$. Since we identify $\mathfrak{t}$ with $\mathfrak{t}^*$, we can consider $\mathfrak{t}$-compatible parabolic subalgebras $\mathfrak{p}_\lambda$ for $\lambda \in \mathfrak{t}^*$.

By $\mathfrak{m}_\lambda$ and $\mathfrak{n}_\lambda$ we denote respectively the reductive part of $\mathfrak{p}_\lambda$ (containing $\mathfrak{h}$) and the nilradical of $\mathfrak{p}_\lambda$. A $\mathfrak{t}$-compatible parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \supsetplus \mathfrak{n}$ (i.e., $\mathfrak{p} = \mathfrak{p}_\lambda$ for some $\lambda \in \mathfrak{t}^*$) is *minimal* if it does not properly contain another $\mathfrak{t}$-compatible parabolic subalgebra. It is easy to see that a $\mathfrak{t}$-compatible parabolic subalgebra $\mathfrak{p}_\lambda$ is minimal if and only if $\mathfrak{m}_\lambda$ equals the centralizer $C_\mathfrak{g}(\mathfrak{t})$, or equivalently if and only if λ is $(\mathfrak{g},\mathfrak{k})$-regular.

A $\mathfrak{k}$-*type* is by definition a simple finite-dimensional $\mathfrak{k}$-module. By V_μ we will denote a $\mathfrak{k}$-type with $\mathfrak{b}_\mathfrak{k}$-highest weight μ (μ is then $\mathfrak{k}$-integral and $\mathfrak{b}_\mathfrak{k}$-dominant). If M is a $(\mathfrak{g},\mathfrak{k})$-module and V_μ is a $\mathfrak{k}$-type, let $M[\mu]$ denote the V_μ-isotypic $\mathfrak{k}$-submodule of M. (See the discussion after (6) in Sect. 1.) Let V_μ be a $\mathfrak{k}$-type such that $\mu + 2\rho$ is $(\mathfrak{g},\mathfrak{k})$-regular, and let $\mathfrak{p} = \mathfrak{m} \supsetplus \mathfrak{n}$ be the $\mathfrak{t}$-compatible parabolic subalgebra $\mathfrak{p}_{\mu+2\rho}$. Note that $\mathfrak{p}$ is a minimal $\mathfrak{t}$-compatible parabolic subalgebra. Put $\rho_\mathfrak{n} := \rho_{\mathrm{ch}_\mathfrak{t}\mathfrak{n}}$.

The following is a key definition. We say that V_μ is *generic* if the following two conditions hold:

(1) $\langle\mathrm{Re}\,\mu + 2\rho - \rho_\mathfrak{n}, \alpha\rangle \ge 0$ for every $\mathfrak{t}$-weight α of $\mathfrak{n}_\mathfrak{k}$.
(2) $\langle\mathrm{Re}\,\mu + 2\rho - \rho_S, \rho_S\rangle > 0$ for every submultiset S of $\mathrm{ch}_\mathfrak{t}\mathfrak{n}$.

One can show that the following is a sufficient condition for genericity: $|\langle\mathrm{Re}\,\mu + 2\rho, \alpha\rangle| \ge c$ for any $\mathfrak{t}$-weight α of $\mathfrak{g}$ and a suitably large positive constant c, depending only on the pair $(\mathfrak{g},\mathfrak{k})$.

Let $\Theta_{\mathfrak{k}}$ be the discrete subgroup of $Z(\mathfrak{k})^*$ generated by $\mathrm{supp}_{Z(\mathfrak{k})}\,\mathfrak{g}$. By $\mathcal{M}$ we denote the class of $(\mathfrak{g},\mathfrak{k})$-modules M for which there exists a finite subset $S \subset Z(\mathfrak{k})^*$ such that $\mathrm{supp}_{Z(\mathfrak{k})}\, M \subset (S + \Theta_{\mathfrak{k}})$. Note that any finite-length $(\mathfrak{g},\mathfrak{k})$-module lies in the class $\mathcal{M}$.

If M is a module in $\mathcal{M}$, a $\mathfrak{k}$-type V_μ of M is *minimal* if the function $\mu' \mapsto \|\operatorname{Re}\mu' + 2\rho\|^2$ defined on the set $\{\mu' \in \mathfrak{t}^* \mid M[\mu'] \neq 0\}$ has a minimum at μ. Any nonzero $(\mathfrak{g},\mathfrak{k})$-module M in $\mathcal{M}$ has a minimal $\mathfrak{k}$-type. This follows from the fact that the squared length of a vector has a minimum on every shifted lattice in Euclidean space.

We need also the following "production" or "coinduction" functor (see [Bla]) from the category of $(\mathfrak{p},\mathfrak{t})$-modules to the category of $(\mathfrak{g},\mathfrak{t})$-modules:

$$\mathrm{pro}_{\mathfrak{p},\mathfrak{t}}^{\mathfrak{g},\mathfrak{t}}(N) := \Gamma_{\mathfrak{t},0}\big(\mathrm{Hom}_{U\mathfrak{p}}(U\mathfrak{g}, N)\big).$$

The functor $\mathrm{pro}_{\mathfrak{p},\mathfrak{t}}^{\mathfrak{g},\mathfrak{t}}$ is exact.

Definition 3.1 Let $\mathfrak{p} = \mathfrak{m} \supsetplus \mathfrak{n}$ be a minimal $\mathfrak{t}$-compatible parabolic subalgebra, and E be a simple finite-dimensional $\mathfrak{p}$-module on which $\mathfrak{t}$ acts via a weight ω. We call the series of $(\mathfrak{g},\mathfrak{k})$-modules of finite type

$$F^*(\mathfrak{p}, E) := R^*\Gamma_{\mathfrak{k},\mathfrak{t}}\big(\mathrm{pro}_{\mathfrak{p},\mathfrak{t}}^{\mathfrak{g},\mathfrak{t}}\big(E \otimes_{\mathbb{C}} \Lambda^{\dim\mathfrak{n}}(\mathfrak{n})\big)\big)$$

the fundamental series of generalized Harish-Chandra modules.

Set $\mu := \omega + 2\rho_{\mathfrak{n}}^{\perp}$. It is proved in [PZ2] that the following assertions hold under the assumptions that $\mathfrak{p} \subseteq \mathfrak{p}_{\mu+2\rho}$ and that μ is $\mathfrak{b}_{\mathfrak{k}}$-dominant and $\mathfrak{k}$-integral.

(a) $F^*(\mathfrak{p}, E)$ is a $(\mathfrak{g},\mathfrak{k})$-module of finite type in the class $\mathcal{M}$.
(b) There is a $\mathfrak{k}$-module isomorphism

$$F^s(\mathfrak{p}, E)[\mu] \cong \mathbf{C}^{\dim E} \otimes_{\mathbb{C}} V_\mu,$$

and V_μ is the unique minimal $\mathfrak{k}$-type of $F^s(\mathfrak{p}, E)$.
(c) Let $\bar{F}^s(\mathfrak{p}, E)$ be the $\mathfrak{g}$-submodule of $F^s(\mathfrak{p}, E)$ generated by $F^s(\mathfrak{p}, E)[\mu]$. Then any simple quotient of $\bar{F}^s(\mathfrak{p}, E)$ has minimal $\mathfrak{k}$-type V_μ.

The following theorems provide the basis of the classification of $(\mathfrak{g},\mathfrak{k})$-modules with generic minimal $\mathfrak{k}$-type. The classification is then stated as a corollary.

Theorem 3.2 (First reconstruction theorem, [PZ3]) *Let M be a simple $(\mathfrak{g},\mathfrak{k})$-module of finite type with a minimal $\mathfrak{k}$-type V_μ which is generic. Then $\mathfrak{p} := \mathfrak{p}_{\mu+2\rho} = \mathfrak{m} \supsetplus \mathfrak{n}$ is a minimal $\mathfrak{t}$-compatible parabolic subalgebra. Let E be the $\mathfrak{p}$-module $H^r(\mathfrak{n}, M)(\mu - 2\rho_{\mathfrak{n}}^{\perp})$ with trivial $\mathfrak{n}$-action, where $r = \dim(\mathfrak{n} \cap \mathfrak{k}^{\perp})$. Then E is a simple $\mathfrak{p}$-module, and M is canonically isomorphic to $\bar{F}^s(\mathfrak{p}, E)$ for $s = \dim(\mathfrak{n} \cap \mathfrak{k})$.*

Theorem 3.3 (Second reconstruction theorem, [PZ3]) *Assume that the pair $(\mathfrak{g},\mathfrak{k})$ is regular, i.e., $\mathfrak{t}$ contains a regular element of $\mathfrak{g}$. Let M be a simple $(\mathfrak{g},\mathfrak{k})$-module (a priori of infinite type) with a minimal $\mathfrak{k}$-type V_μ which is generic. Then M has finite type, and hence by Theorem 3.2, M is canonically isomorphic to $\bar{F}^s(\mathfrak{p}, E)$ (where $\mathfrak{p}$, E, and s are as in Theorem 3.2).*

Corollary 3.4 (Classification of generalized Harish-Chandra modules with generic minimal $\mathfrak{k}$-type) *Fix a generic $\mathfrak{k}$-type V_μ. The simple $(\mathfrak{g}, \mathfrak{k})$-modules of finite type whose minimal $\mathfrak{k}$-type is isomorphic to V_μ are in a natural bijective correspondence with the simple finite-dimensional $\mathfrak{p}_{\mu+2\rho}$-modules on which $\mathfrak{t}$ acts via $\mu - 2\rho_{\mathfrak{n}}^{\perp}$.*

Note that if $\mathrm{rk}\mathfrak{k} = \mathrm{rk}\mathfrak{g}$, there exists a unique simple finite-dimensional $\mathfrak{p}_{\mu+2\rho}$-module E on which $\mathfrak{t}$ acts via $\mu - 2\rho_{\mathfrak{n}}^{\perp}$. If $\mathrm{rk}\mathfrak{g} - \mathrm{rk}\mathfrak{k} = d > 0$, there exists a d-parameter family of such $\mathfrak{p}_{\mu+2\rho}$-modules.

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Part II
Papers

B-Orbits of 2-Nilpotent Matrices and Generalizations

Magdalena Boos and Markus Reineke

Abstract The orbits of the group B_n of upper-triangular matrices acting on 2-nilpotent complex matrices via conjugation are classified via oriented link patterns, generalizing Melnikov's classification of the B_n-orbits on upper-triangular such matrices. The orbit closures and the "building blocks" of minimal degenerations of orbits are described. The classification uses the theory of representations of finite-dimensional algebras. Furthermore, we initiate the study of the B_n-orbits on arbitrary nilpotent matrices.

Keywords Borel orbits in nilpotent matrices · Auslander–Reiten quiver · Orbit closure relation · Semiinvariants

Mathematics Subject Classification (2010) 14L30 · 16G70

1 Introduction

The study of adjoint actions and variants thereof, and in particular the classification of orbits for such actions and the description of the orbit closures, are a common theme in Lie representation theory. The archetypical example is the Jordan–Gerstenhaber theory for the conjugacy classes of complex $n \times n$-matrices.

A recent special case is Melnikov's study of the action of the Borel subgroup B_n on upper-triangular 2-nilpotent matrices via conjugation [8, 9]. The orbits and their closures are described there combinatorially in terms of so-called link patterns, which we will recapitulate in Sect. 2.1.

Our aim in this paper is to generalize the work of Melnikov by extending the variety of upper-triangular 2-nilpotent matrices to all 2-nilpotent matrices. The basic setup to reach this goal is a translation of the classification problem to a problem in representation theory of finite-dimensional algebras. More precisely, this translation yields a bijection between the orbits and the isomorphism classes of certain representations of a specific finite-dimensional algebra, see Sect. 3.1. After a brief

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summary of methods from the representation theory of algebras in Sect. 2.2 (see, for example, [1]), we calculate all indecomposable representations using Auslander–Reiten theory [2] in Sect. 3.2 and classify the required representations. This gives a combinatorial classification in terms of oriented link patterns in Sect. 3.3. In fact, our method generalizes to the conjugation action of a parabolic subgroup $\mathcal{P}$ containing B_n on $\mathcal{N}_n^{(2)}$; see Proposition 3.6.

Since several results on orbit closures for representations of finite-dimensional algebras are available through work of Zwara [10, 11], we can also characterize the orbit closures of 2-nilpotent matrices in Sect. 4.

Finally, we study the conjugation action of upper-triangular matrices on arbitrary nilpotent matrices. We provide a generic normal form for the orbits of this action in Sect. 5.1 and construct a large class of semiinvariants in Sect. 5.2.

2 The Basic Setup

In this section, we fix some notation and collect information about the aforementioned group action. In addition, we summarize material from the representation theory of finite-dimensional algebras.

Let $k = \mathbf{C}$ be the field of complex numbers. We denote by $B_n \subset \mathrm{GL}_n(k)$ the Borel subgroup of upper-triangular matrices, by $\mathcal{N}_n \subset M_{n\times n}(k)$ the variety of nilpotent $n \times n$-matrices N, and by $\mathcal{N}_n^{(2)}$ the closed subvariety of 2-nilpotent such matrices, that is, $N^2 = 0$. Obviously, $\mathrm{GL}_n(k)$ and B_n act on $\mathcal{N}_n$ and on $\mathcal{N}_n^{(2)}$ via conjugation.

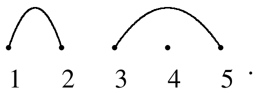
In case of the action of $\mathrm{GL}_n(k)$ on $\mathcal{N}_n$, the classical Jordan–Gerstenhaber theory gives a complete classification of the orbits and their closures in terms of partitions (or, equivalently, Young diagrams).

Our aim is to classify the orbits $\mathcal{O}_A$ of 2-nilpotent matrices $A \in \mathcal{N}_n^{(2)}$ under the action of B_n. Such a classification will be given in terms of *oriented link patterns*; these are oriented graphs on the set of vertices $\{1,\ldots,n\}$ such that every vertex is incident with at most one arrow. This is followed by a description of the orbit closures by giving a necessary and sufficient condition to decide whether one orbit is contained in the closure of another and by a method to construct all orbits contained in a given orbit closure. These descriptions are also given in terms of oriented link patterns.

2.1 Results of Melnikov

The group B_n also acts on $\mathfrak{n}_n \subset \mathcal{N}_n$, the space of all upper-triangular matrices in $\mathcal{N}_n$, and on $\mathfrak{n}_n^{(2)} = \mathfrak{n}_n \cap \mathcal{N}_n^{(2)}$. The orbits and their closures for the latter action are described by Melnikov in [8, 9]. Since these results will be generalized in the following, we describe them in more detail.

Let $S_n^{(2)}$ be the set of involutions in the symmetric group S_n in n letters. An element σ of $S_n^{(2)}$ is represented by a so-called link pattern, an unoriented graph with vertices $\{1, \ldots, n\}$ and an edge between i and j if $\sigma(i) = j$. For example, the involution $(1, 2)(3, 5) \in S_5$ corresponds to the link pattern



For $\sigma \in S_n^{(2)}$, define $N_\sigma \in \mathcal{N}_n^{(2)}$ by

$$(N_\sigma)_{i,j} = \begin{cases} 1 & \text{if } i < j \text{ and } \sigma(i) = j, \\ 0 & \text{otherwise,} \end{cases}$$

and denote by $\mathcal{B}_\sigma = B_n \cdot N_\sigma$ the B_n-orbit of N_σ.

Theorem 2.1 [8] *Every orbit of B_n in $\mathfrak{n}_n^{(2)}$ is of the form $\mathcal{B}_\sigma$ for a unique $\sigma \in S_n^{(2)}$.*

The next step is to look at the (Zariski-)closures of the orbits $\mathcal{B}_\sigma$.

For $1 \leq i < j \leq n$, consider the canonical projection $\pi_{i,j} : \mathfrak{n}_n^{(2)} \to \mathfrak{n}_{j-i+1}^{(2)}$ corresponding to the deletion of the first $i - 1$ and the last $n - j$ columns and rows of a matrix in $\mathfrak{n}_n^{(2)}$. Define the rank matrix R_u of $u \in \mathfrak{n}_n^{(2)}$ by

$$(R_u)_{i,j} = \begin{cases} \operatorname{rank}(\pi_{i,j}(u)) & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases}$$

The rank matrix R_u is B_n-invariant, and we denote $R_\sigma = R_{N_\sigma}$ for $\sigma \in S_n^{(2)}$. We define a partial ordering on the set of rank matrices by $R_{\sigma'} \preccurlyeq R_\sigma$ if $(R_{\sigma'})_{i,j} \leq (R_\sigma)_{i,j}$ for all i and j, inducing a partial ordering on $S_n^{(2)}$ by $\sigma' \preccurlyeq \sigma$ if $R_{\sigma'} \preccurlyeq R_\sigma$.

Theorem 2.2 [9] *The orbit closure of $\mathcal{B}_\sigma$ is given by $\overline{\mathcal{B}_\sigma} = \bigcup_{\sigma' \preccurlyeq \sigma} \mathcal{B}_{\sigma'}$. Moreover, the entry $(R_\sigma)_{i,j}$ of the rank matrix equals the number of edges with end points e_1 and e_2 such that $i \leq e_1, e_2 \leq j$ in the link pattern of σ.*

The theorems thus give a combinatorial characterization of the B_n-orbits in $\mathfrak{n}_n^{(2)}$ and their orbit closures in terms of link patterns.

2.2 Representations of Algebras

As we make key use of results from the representation theory of finite-dimensional algebras for the study of the action of B_n on $\mathcal{N}_n^{(2)}$, we now recall the basic setup of this theory and refer to [1] and [2] for a thorough treatment. Let $\mathcal{Q}$ be a finite quiver, that is, a directed graph $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, t)$ consisting of a finite set of vertices $\mathcal{Q}_0$ and a finite set of arrows $\mathcal{Q}_1$, whose elements are written as $\alpha : s(\alpha) \to t(\alpha)$; the vertices $s(\alpha)$ and $t(\alpha)$ are called the source and the target of α, respectively. A path in $\mathcal{Q}$ is a sequence of arrows $\omega = \alpha_s \ldots \alpha_1$ such that $t(\alpha_k) = s(\alpha_{k+1})$ for all $k =$

$1, \dots, s-1$; we formally include a path ε_i of length zero for each $i \in \mathcal{Q}_0$ starting and ending in i. We have an obvious notion of concatenation $\omega\omega'$ of paths $\omega = \alpha_s \dots \alpha_1$ and $\omega' = \beta_t \dots \beta_1$ such that $t(\beta_t) = s(\alpha_1)$.

The path algebra $k\mathcal{Q}$ is defined as the k-vector space with basis consisting of all paths in $\mathcal{Q}$, and with multiplication

$$\omega \cdot \omega' = \begin{cases} \omega\omega' & \text{if } t(\beta_t) = s(\alpha_1), \\ 0 & \text{otherwise.} \end{cases}$$

The radical $\mathrm{rad}(k\mathcal{Q})$ is defined as the (two-sided) ideal generated by paths of positive length. An ideal I of $k\mathcal{Q}$ is called admissible if $\mathrm{rad}(k\mathcal{Q})^s \subset I \subset \mathrm{rad}(k\mathcal{Q})^2$ for some s.

The key feature of such pairs $(\mathcal{Q}, I)$ consisting of a quiver $\mathcal{Q}$ and an admissible ideal $I \subset k\mathcal{Q}$ is the following: every finite-dimensional k-algebra A is Morita-equivalent to an algebra of the form $k\mathcal{Q}/I$, in the sense that their categories of finite-dimensional k-representations are (k-linearly) equivalent.

A finite-dimensional k-representation M of $\mathcal{Q}$ consists of a tuple of k-vector spaces M_i for $i \in \mathcal{Q}_0$ and a tuple of k-linear maps $M_\alpha : M_i \to M_j$ indexed by the arrows $\alpha : i \to j$ in $\mathcal{Q}_1$. A morphism of two such representations $M = ((M_i)_{i\in\mathcal{Q}_0}, (M_\alpha)_{\alpha\in\mathcal{Q}_1})$ and $N = ((N_i)_{i\in\mathcal{Q}_0}, (N_\alpha)_{\alpha\in\mathcal{Q}_1})$ consists of a tuple of k-linear maps $(f_i : M_i \to N_i)_{i\in\mathcal{Q}_0}$ such that

$$f_j M_\alpha = N_\alpha f_i \quad \text{for all } \alpha : i \to j \text{ in } \mathcal{Q}_1.$$

For a representation M and a path ω in $\mathcal{Q}$ as above, we denote $M_\omega = M_{\alpha_s} \cdots M_{\alpha_1}$. We call M bound by I if $\sum_\omega \lambda_\omega M_\omega = 0$ whenever $\sum_\omega \lambda_\omega \omega \in I$.

The abelian k-linear category of all representations of $\mathcal{Q}$ bound by I is denoted by $\mathrm{rep}_k(\mathcal{Q}, I)$; it is equivalent to the category of finite-dimensional representations of the algebra $k\mathcal{Q}/I$. We have thus found a "linear algebra model" for the category of finite-dimensional representations of an arbitrary finite-dimensional k-algebra A.

We define the dimension vector $\underline{\dim}\, M \in \mathbf{N}\mathcal{Q}_0$ of M by $(\underline{\dim}\, M)_i = \dim_k M_i$ for $i \in \mathcal{Q}_0$. For a fixed dimension vector $\underline{d} \in \mathbf{N}\mathcal{Q}_0$, we consider the affine space $R_{\underline{d}}(Q) = \bigoplus_{\alpha:i\to j} \mathrm{Hom}_k(k^{d_i}, k^{d_j})$; its points naturally correspond to representations M of $\mathcal{Q}$ of dimension vector $\underline{d}$ with $M_i = k^{d_i}$ for $i \in \mathcal{Q}_0$. Via this correspondence, the set of such representations bound by I corresponds to a closed subvariety $R_{\underline{d}}(Q, I) \subset R_{\underline{d}}(Q)$. It is obvious that the algebraic group $\mathrm{GL}_{\underline{d}} = \prod_{i\in\mathcal{Q}_0} \mathrm{GL}_k(k^{d_i})$ acts on $R_{\underline{d}}(Q)$ and on $R_{\underline{d}}(Q, I)$ via the base change $(g_i)_i \cdot (M_\alpha)_\alpha = (g_j M_\alpha g_i^{-1})_{\alpha:i\to j}$. By definition, the $\mathrm{GL}_{\underline{d}}$-orbits $\mathcal{O}_M$ of this action naturally correspond to the isomorphism classes of representations M in $\mathrm{rep}_k(\mathcal{Q}, I)$ of dimension vector $\underline{d}$.

By the Krull–Schmidt theorem, every representation in $\mathrm{rep}_k(\mathcal{Q}, I)$ is isomorphic to a direct sum of indecomposables, unique up to isomorphisms and permutations. Thus, knowing the isomorphism classes of indecomposable representations in $\mathrm{rep}_k(\mathcal{Q}, I)$ and their dimension vectors, we can classify the orbits of $\mathrm{GL}_{\underline{d}}$ in $R_{\underline{d}}(\mathcal{Q}, I)$.

For certain classes of finite-dimensional algebras, a convenient tool for the classification of the indecomposable representations is the Auslander–Reiten quiver $\Gamma(\mathcal{Q}, I)$ of $k\mathcal{Q}/I$. Its vertices $[X]$ are given by the isomorphism classes of indecomposable representations of $k\mathcal{Q}/I$; the arrows between two such vertices $[X]$ and $[Y]$ are parameterized by a basis of the space of so-called irreducible maps $f : X \to Y$. Several standard techniques are available for the calculation of $\Gamma(\mathcal{Q}, I)$, see, for example, [1] and [7]. We will illustrate one of these techniques, namely the use of covering quivers, in Sect. 3.2 in a situation relevant for our setup.

3 Classification of Orbits

3.1 Translation to a Representation-Theoretic Problem

Our aim in this section is to translate the classification problem for the action of B_n on $\mathcal{N}_n^{(2)}$ into a representation-theoretic one. The following is a well-known fact on associated fiber bundles:

Theorem 3.1 *Let G be an algebraic group, let X and Y be G-varieties, and let $\pi : X \to Y$ be a G-equivariant morphism. Assume that Y is a single G-orbit, $Y = Gy_0$. Let H be the stabilizer of y_0 and set $F := \pi^{-1}(y_0)$. Then X is isomorphic to the associated fiber bundle $G \times^H F$, and the embedding $\phi : F \hookrightarrow X$ induces a bijection between H-orbits in F and G-orbits in X preserving orbit closures.*

We consider the following quiver, denoted by $\mathcal{Q}$ from now on,

$$\mathcal{Q}: \quad \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \quad \cdots \quad \underset{n-2}{\bullet} \xrightarrow{\alpha_{n-2}} \underset{n-1}{\bullet} \xrightarrow{\alpha_{n-1}} \underset{n}{\bullet} \circlearrowleft \alpha$$

together with the ideal $I \subset k\mathcal{Q}$ generated by the path α^2. We consider the full subcategory $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ of $\mathrm{rep}_k(\mathcal{Q}, I)$ consisting of representations M for which the linear maps $M_{\alpha_1}, \ldots, M_{\alpha_{n-1}}$ are injective. Corresponding to this subcategory, we have an open subset $R_{\underline{d}}^{\mathrm{inj}}(\mathcal{Q}, I) \subset R_{\underline{d}}(\mathcal{Q}, I)$, which is stable under the $\mathrm{GL}_{\underline{d}}$-action. We consider the dimension vector $\underline{d_0} := (1, 2, \ldots, n) \in \mathbf{N}^n$.

Lemma 3.2 *There exists a closure-preserving bijection Φ between the set of B_n-orbits in $\mathcal{N}_n^{(2)}$ and the set of $\mathrm{GL}_{\underline{d_0}}$-orbits in $R_{\underline{d_0}}^{\mathrm{inj}}(\mathcal{Q}, I)$.*

Proof Consider the subquiver $\widetilde{\mathcal{Q}}$ of $\mathcal{Q}$ with $\widetilde{\mathcal{Q}}_0 = \mathcal{Q}_0$ and $\widetilde{\mathcal{Q}}_1 = \mathcal{Q}_1 \setminus \{\alpha\}$. We have a natural $\mathrm{GL}_{\underline{d_0}}$-equivariant projection $\pi : R_{\underline{d_0}}^{\mathrm{inj}}(\mathcal{Q}, I) \to R_{\underline{d_0}}^{\mathrm{inj}}(\widetilde{\mathcal{Q}})$. The variety $R_{\underline{d_0}}^{\mathrm{inj}}(\widetilde{\mathcal{Q}})$ consists of tuples of injective maps. Thus, the action of $\mathrm{GL}_{\underline{d_0}}$ on $R_{\underline{d_0}}(\widetilde{\mathcal{Q}})$ is easily seen to be transitive. Namely, $R_{\underline{d_0}}^{\mathrm{inj}}(\widetilde{\mathcal{Q}})$ is the orbit of the representation

$$y_0 := k \xrightarrow{\iota_1} k^2 \xrightarrow{\iota_2} \cdots \xrightarrow{\iota_{n-2}} k^{n-1} \xrightarrow{\iota_{n-1}} k^n$$

with ι_i being the canonical embedding from k^i to k^{i+1}. The stabilizer H of y_0 is isomorphic to B_n, and the fiber of π over y_0 is isomorphic to $\mathcal{N}_n^{(2)}$. Thus, $R_{\underline{d_0}}^{\mathrm{inj}}(\mathcal{Q}, I)$ is isomorphic to the associated fiber bundle $\mathrm{GL}_{\underline{d_0}} \times^{B_n} \mathcal{N}_n^{(2)}$, yielding the claimed bijection. □

3.2 *Classification of Indecomposables in* $\mathbf{rep}_k(\mathcal{Q}, I)$

By the results of the previous section, it suffices to classify the indecomposable representations in $\mathrm{rep}_k(\mathcal{Q}, I)$ to obtain a classification of the orbits of B_n in $\mathcal{N}_n^{(2)}$. We compute the Auslander–Reiten quiver Γ of $k\mathcal{Q}/I$ using covering theory, which is described in [7], as mentioned before. We consider the (infinite) quiver $\widehat{\mathcal{Q}}$ given by

$$
\begin{array}{lcccccccccccc}
 & & & & & & \vdots & & & & & & \\
 & & & & & & & & & & & & \downarrow \\
\widehat{\mathcal{Q}}: & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \\
 & & & & & & & & & & & & \downarrow \alpha_i \\
 & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \\
 & & & & & & & & & & & & \downarrow \alpha_{i+1} \\
 & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \\
 & & & & & & & & & & & & \downarrow \\
 & & & & & & \vdots & & & & & & \\
 & 1 & & 2 & & 3 & & n-2 & & n-1 & & n &
\end{array}
$$

with the ideal $\widehat{I}$ generated by all paths $\alpha_{i+1}\alpha_i$, and the quiver $\mathcal{Q}'$ given by

$$
\begin{array}{lcccccccccccc}
\mathcal{Q}': & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \\
 & & & & & & & & & & & & \downarrow \alpha \\
 & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \\
 & 1 & & 2 & & 3 & & n-2 & & n-1 & & n &
\end{array}
$$

The quiver $\widehat{\mathcal{Q}}$ carries a natural action of the group $\mathbf{Z}$ by shifting the rows, so that $\widehat{\mathcal{Q}}/\mathbf{Z} \cong \mathcal{Q}$. Moreover, $\mathcal{Q}'$ naturally embeds into $\widehat{\mathcal{Q}}$ so that the composition of this inclusion with the projection $\widehat{\mathcal{Q}} \to \mathcal{Q}$ is surjective. By results of covering theory [7], we have corresponding maps of the Auslander–Reiten quivers, namely an embedding $\Gamma(\mathcal{Q}') \to \Gamma(\widehat{\mathcal{Q}}, \widehat{I})$ and a quotient $\Gamma(\widehat{\mathcal{Q}}, \widehat{I}) \to \Gamma(\mathcal{Q}, I)$, such that the composition is surjective. Since $\mathcal{Q}'$ is nothing else than a Dynkin quiver of type A_{2n}, it is routine to calculate its Auslander–Reiten quiver (see [1]), and we derive the Auslander–Reiten quiver $\Gamma = \Gamma(\mathcal{Q}, I)$ just by making the identifications resulting from the action of $\mathbf{Z}$, which can be read off from the dimension vectors of indecomposable representations. More examples and details concerning the calculation of Auslander–Reiten quivers using covering theory can also be found in [7].

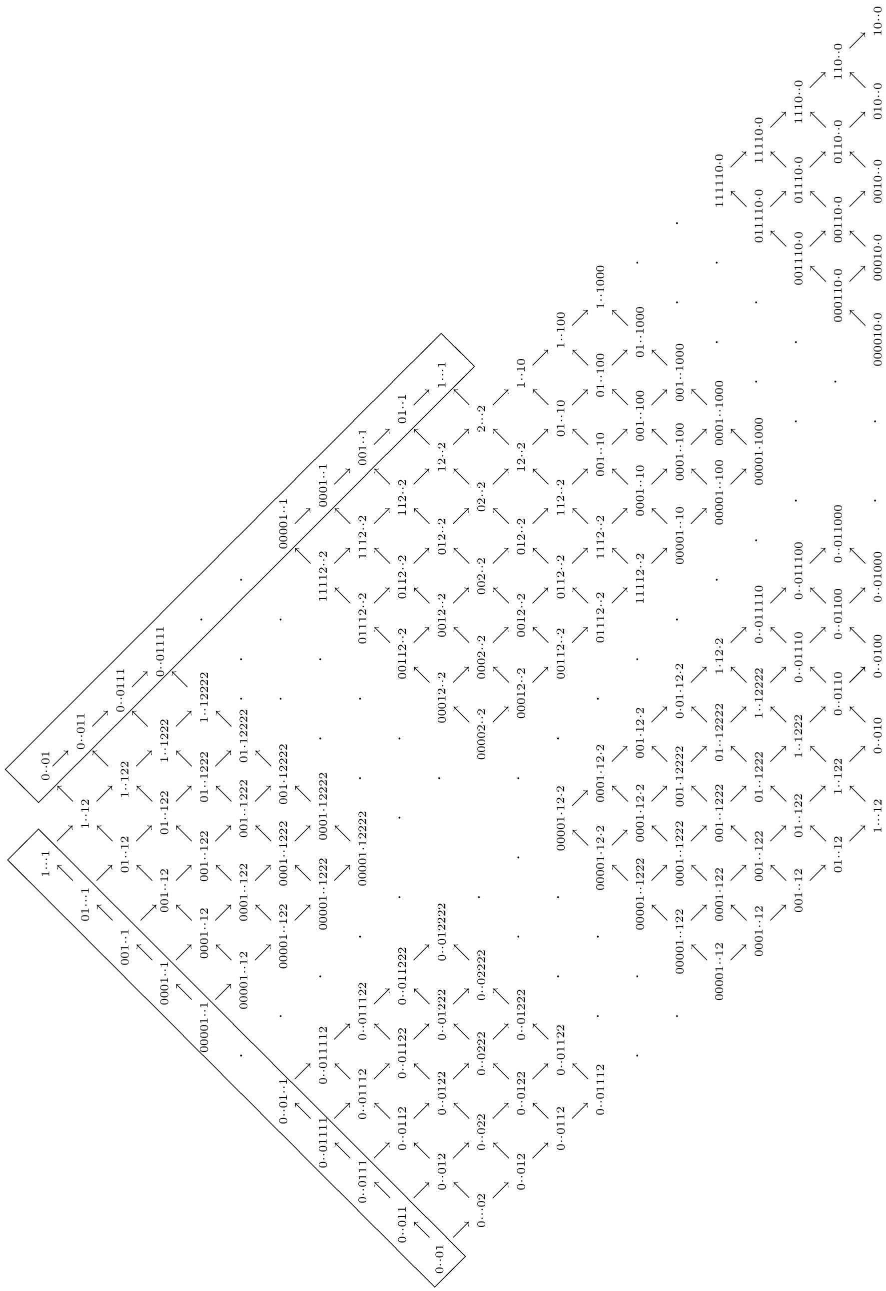


Fig. 1 The Auslander–Reiten quiver of $\mathrm{rep}_k(\mathcal{Q}, I)$

We finally arrive at the picture given in Fig. 1. Note that the dimension vectors in the boxed regions appear twice; they have to be identified, since they correspond to a unique isomorphism class of indecomposable representations.

We define the following representations $\mathcal{U}_{i,j}$ for $1 \le i, j \le n$, $\mathcal{V}_i$ for $1 \le i \le n$ and $\mathcal{W}_{i,j}$ for $1 \le i \le j \le n$ in $\mathrm{rep}_k(\mathcal{Q}, I)$ (graphically represented by dots for basis elements and arrows for a map sending one basis element to another one):

$\mathcal{U}_{i,j}$ for $1 \le j \le i \le n$:

$$\begin{array}{ccccccccccccccccc}
0 & \overset{0}{\to} & \cdots & \overset{0}{\to} & 0 & \overset{0}{\to} & k & \overset{id}{\to} & \cdots & \overset{id}{\to} & k & \overset{e_1}{\to} & k^2 & \overset{id}{\to} & \cdots & \overset{id}{\to} & k^2 \circlearrowleft \alpha \\
 & & & & & & \bullet & \to & \cdots & \to & \bullet & \to & \bullet & \to & \cdots & \to & \bullet \\
 & & & & & & j & & & & & & i & & & & n \\
 & & & & & & & & & & & & \bullet & \to & \cdots & \to & \bullet
\end{array}$$

$\mathcal{U}_{i,j}$ for $1 \le i < j \le n$:

$$\begin{array}{ccccccccccccccccc}
0 & \overset{0}{\to} & \cdots & \overset{0}{\to} & 0 & \overset{0}{\to} & k & \overset{id}{\to} & \cdots & \overset{id}{\to} & k & \overset{e_2}{\to} & k^2 & \overset{id}{\to} & \cdots & \overset{id}{\to} & k^2 \circlearrowleft \alpha \\
 & & & & & & \bullet & \to & \cdots & \to & \bullet & \to & \bullet & \to & \cdots & \to & \bullet \\
 & & & & & & i & & & & & & j & & & & n \\
 & & & & & & & & & & & & \bullet & \to & \cdots & \to & \bullet
\end{array}$$

$\mathcal{V}_i$ for $1 \le i \le n$:

$$\begin{array}{ccccccccccc}
0 & \overset{0}{\to} & \cdots & \overset{0}{\to} & 0 & \overset{0}{\to} & k & \overset{id}{\to} & \cdots & \overset{id}{\to} & k \circlearrowleft 0 \\
 & & & & & & i & & & & n \\
 & & & & & & \bullet & \to & \cdots & \to & \bullet
\end{array}$$

$\mathcal{W}_{i,j}$ for $1 \le i \le j < n$:

$$\begin{array}{ccccccccccccccccc}
0 & \overset{0}{\to} & \cdots & \overset{0}{\to} & 0 & \overset{0}{\to} & k & \overset{id}{\to} & \cdots & \overset{id}{\to} & k & \overset{0}{\to} & 0 & \overset{0}{\to} & \cdots & \overset{0}{\to} & 0 \circlearrowleft 0 \\
 & & & & & & i & & & & j & & & & & & n \\
 & & & & & & \bullet & \to & \cdots & \to & \bullet
\end{array}$$

Here we denote $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$ and $\alpha = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

Theorem 3.3 *The representations $\mathcal{U}_{i,j}$, $\mathcal{V}_i$ and $\mathcal{W}_{i,j}$ form a system of representatives of the indecomposable objects in* $\mathrm{rep}_k(\mathcal{Q}, I)$. *The representations $\mathcal{U}_{i,j}$ and $\mathcal{V}_i$ form a system of representatives of the indecomposable objects in* $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$.

Proof The endomorphism rings of these representations are easily computed to be

$$\mathrm{End}(\mathcal{U}_{i,j}) \cong k \quad \text{for } i > j, \qquad \mathrm{End}(\mathcal{U}_{i,j}) \cong k[x]/(x^2) \quad \text{for } i \le j,$$
$$\mathrm{End}(\mathcal{V}_i) \cong k, \qquad \mathrm{End}(\mathcal{W}_{i,j}) \cong k,$$

and thus they are indecomposable. Their dimension vectors are

$$(0\ldots01\ldots12\ldots2),\quad (0\ldots01\ldots1)\quad\text{and}\quad (0\ldots01\ldots10\ldots0),$$

respectively. These are precisely the dimension vectors appearing in $\Gamma(\mathcal{Q}, I)$, and thus we have found all indecomposables. It is clear from the definition that the indecomposable representations belonging to $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ are the $\mathcal{U}_{i,j}$ and the $\mathcal{V}_i$. □

3.3 *Classification of B_n-Orbits in $\mathcal{N}_n^{(2)}$*

Our next aim is to parameterize the isomorphism classes of representations in $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ of dimension vector $\underline{d_0}$. As mentioned before, the Krull–Schmidt theorem states that every representation can be decomposed into a direct sum of indecomposables in an essentially unique way.

Theorem 3.4 *The isomorphism classes M in $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ of dimension vector $\underline{d_0}$ are in natural bijection to*

(1) *$n \times n$-matrices $A = (m_{i,j})_{i,j}$ with entries 0 or 1, such that $\sum_j m_{i,j} + \sum_j m_{j,i} \leq 1$ for all $i = 1, \ldots, n$,*
(2) *oriented link patterns on $\{1, \ldots, n\}$, that is, oriented graphs on the set $\{1, \ldots, n\}$ such that every vertex is incident with at most one arrow.*

Moreover, if an isomorphism class M corresponds to a matrix A under this bijection, the orbit $\mathcal{O}_M \subset R_{\underline{d_0}}^{\mathrm{inj}}(\mathcal{Q}, I)$ and the orbit $\mathcal{O}_A \in \mathcal{N}_n^{(2)}$ correspond to each other via the bijection Φ of Lemma 3.2.

Proof Let M be a representation in $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ of dimension vector $\underline{d_0}$, so

$$M = \bigoplus_{i,j=1}^{n} \mathcal{U}_{i,j}^{m_{i,j}} \oplus \bigoplus_{i=1}^{n} \mathcal{V}_i^{n_i}$$

for some multiplicities $m_{i,j}, n_i \in \mathbf{N}$ by Theorem 3.3. Since $\underline{\dim}\, M = (1, 2, \ldots, n)$, we simply need to calculate all tuples $(m_{i,j}, n_i)$ such that

$$\sum_{i,j=1}^{n} m_{i,j}\, \underline{\dim}\, \mathcal{U}_{i,j} + \sum_{i=1}^{n} n_i\, \underline{\dim}\, \mathcal{V}_i = \underline{d_0} = (1, 2, \ldots, n).$$

Applying the automorphism δ of $\mathbf{Z}^n$ defined by

$$\delta(d_1, d_2, \ldots, d_n) = (d_1, d_2 - d_1, d_3 - d_2, \ldots, d_n - d_{n-1}),$$

this condition is equivalent to

$$\sum_{i,j=1}^{n} m_{i,j}\delta(\underline{\dim}\,\mathcal{U}_{i,j}) + \sum_{i=1}^{n} n_i\delta(\underline{\dim}\,\mathcal{V}_i) = (1, 1, \ldots, 1, 1).$$

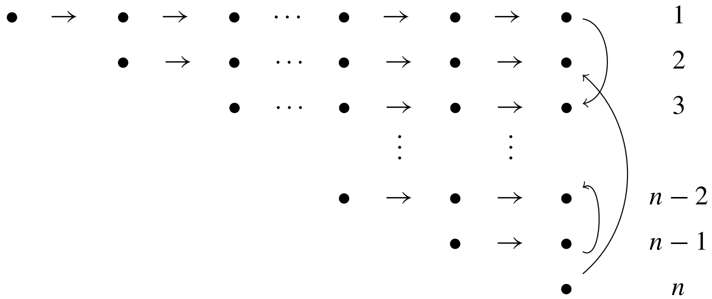
If we fix $i \in \{1, \ldots, n\}$, this condition states that

$$1 = \sum_{j=1}^{n} m_{i,j} + \sum_{j=1}^{n} m_{j,i} + n_i.$$

We can extract an oriented graph on the set of vertices $\{1, \ldots, n\}$ from $(m_{i,j})_{i,j}$ as follows: for all $1 \le i, j \le n$, we have an arrow from j to i if $m_{i,j} = 1$. The conditions on $(m_{i,j})_{i,j}$ ensure that this graph is in fact an oriented link pattern. The matrix $(m_{i,j})_{i,j}$ is obviously 2-nilpotent.

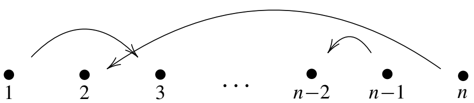
The decomposition of M into indecomposables can be visualized as follows.

M :



The arrows in the rightmost column of the diagram allow us to read off the indecomposable direct summands of M. Namely, $\mathcal{U}_{i,j}$ is a direct summand of M if and only if there is an arrow $j \to i$. If there is no arrow at k, the indecomposable $\mathcal{V}_k$ is a direct summand of M.

Shortening the above picture to the rightmost column, M corresponds to an oriented link pattern:



□

For a given matrix $A \in \mathcal{N}_n^{(2)}$, we would like to decide to which oriented link pattern it corresponds. Define $U_i = \langle e_1, \ldots, e_i\rangle$, the span of the first i coordinate vectors in k^n, and define a matrix $D^A = (d^A_{i,j})_{i,j}$ by setting $d^A_{i,j} := \dim(U_i \cap A(U_j))$ (we formally define $d^A_{i,j} = 0$ for $i = 0$ or $j = 0$). The matrix D^A is obviously an invariant for the B_n-action on $\mathcal{N}_n^{(2)}$. It is easy to extract an oriented link pattern from D^A as follows:

Lemma 3.5 *The matrix A belongs to the orbit of a matrix $(m_{i,j})_{i,j}$ as above if and only if $d^A_{i,j} = \sum_{i' \leq i;\, j' \leq j} m_{i',j'}$ or, conversely, $m_{i,j} = d^A_{i,j} - d^A_{i-1,j} - d^A_{i,j-1} + d^A_{i-1,j-1}$ for all $1 \leq i, j \leq n$.*

Proof By B_n-invariance, we just have to compute D^A for $A = (m_{i,j})_{i,j}$ as in the previous theorem. We have $e_k \in U_i \cap A(U_j)$ if and only $k \leq i$ and there exists $l \leq j$ such that $m_{k,l} = 1$ or, equivalently, such that there exists an arrow $l \to k$ in the corresponding oriented link pattern. Since both U_i and $A(U_j)$ are spanned by coordinate vectors e_k, we thus have $d^A_{i,j} = \dim(U_i \cap A(U_j)) = \sum_{k \leq i,\, l \leq j} m_{k,l}$. The second formula follows. □

Remark We can also rederive Theorem 2.1 of Melnikov: every B_n-orbit of an upper-triangular 2-nilpotent matrix corresponds to the orbit of a representation in $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ of dimension vector $\underline{d_0}$ which does not contain $\mathcal{U}_{i,j}$ for $i \geq j$ as a direct summand. In this case, the corresponding link pattern consists of arrows pointing in the same direction. We can thus delete the orientation and arrive at a link pattern as in [8].

Our method easily generalizes to obtain a classification of orbits for a more general group action: let $\mathcal{P} \subset \mathrm{GL}_n$ be the parabolic subgroup consisting of block-upper triangular matrices with block-sizes $(b_1, \ldots, b_k)$. Then $\mathcal{P}$ acts on $\mathcal{N}_n^{(2)}$ by conjugation, and the same reasoning as above yields a bijection between $\mathcal{P}$-orbits in $\mathcal{N}_n^{(2)}$ and isomorphism classes of representations in $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ of dimension vector $(b_1, b_1 + b_2, \ldots, \sum_{i=1}^{k} b_i)$. Using the analysis of this section, we arrive at the following:

Proposition 3.6 *The $\mathcal{P}$-orbits in $\mathcal{N}_n^{(2)}$ correspond bijectively to $k \times k$-matrices $(m_{i,j})_{i,j}$ such that*

$$\sum_j m_{i,j} + \sum_j m_{j,i} \leq b_i$$

for all $i = 1, \ldots, k$. Consequently, they correspond bijectively to "enhanced oriented link patterns of type $(b_1, \ldots, b_k)$", namely, to oriented graphs on the set $\{1, \ldots, k\}$ such that the vertex i is incident with at most b_i arrows for all i.

4 Orbit Closures

After classifying the orbits via oriented link patterns, we describe the corresponding orbit closures. Again, we will solve this problem using results about the geometry of representations of algebras. Two theorems of Zwara are the key to calculating these orbit closures, see [10] and [11] for more details.

4.1 A Criterion for Degenerations

Let M and M' be two representations in $\mathrm{rep}_k(\mathcal{Q}, I)$ of the same dimension vector $\underline{d}$. We say that M degenerates to M' if $\mathcal{O}_{M'} \subset \overline{\mathcal{O}_M}$ in $R_{\underline{d}}(\mathcal{Q}, I)$, which will be denoted by $M \leq_{\deg} M'$. Since the correspondence Φ of Lemma 3.2 preserves orbit closure relations, we know that $M \leq_{\deg} M'$ for representations M, M' in $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ of dimension vector $\underline{d_0}$ if and only if the corresponding 2-nilpotent matrices $A = (m_{i,j})_{i,j}$ and $A' = (m'_{i,j})_{i,j}$, respectively, fulfill $\mathcal{O}_{A'} \subset \overline{\mathcal{O}_A}$ in $\mathcal{N}_n^{(2)}$.

Theorem 4.1 (*Zwara*) *Suppose that an algebra $k\mathcal{Q}/I$ is representation-finite, that is, $k\mathcal{Q}/I$ admits only finitely many isomorphism classes of indecomposable representations. Let M and M' be two finite-dimensional representations of $k\mathcal{Q}/I$ of the same dimension vector.*

Then $M \leq_{\deg} M'$ if and only if $\dim_k \mathrm{Hom}(U, M) \leq \dim_k \mathrm{Hom}(U, M')$ *for every representation U of $k\mathcal{Q}/I$.*

To simplify notation, we set $[U, V] := \dim_k \mathrm{Hom}(U, V)$ for two representations U and V. Since the dimension of a homomorphism space is additive with respect to direct sums, we only have to consider the inequality $[U, M] \leq [U, M']$ for indecomposable representations U to characterize a degeneration $M \leq_{\deg} M'$. Furthermore, since $[\mathcal{W}_{i,j}, M] = 0$ for all representations M in $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ by a direct calculation, we can restrict these indecomposables U to those of type $\mathcal{U}_{i,j}$ and $\mathcal{V}_i$ of the previous section.

We can easily calculate the dimensions of homomorphism spaces between these indecomposable representations.

Lemma 4.2 *For $i, j, k, l \in \{1, \ldots, n\}$, we have*

- $[\mathcal{V}_k, \mathcal{V}_i] = \delta_{i \leq k}$,
- $[\mathcal{V}_k, \mathcal{U}_{i,j}] = \delta_{i \leq k}$,
- $[\mathcal{U}_{k,l}, \mathcal{V}_i] = \delta_{i \leq l}$,
- $[\mathcal{U}_{k,l}, \mathcal{U}_{i,j}] = \delta_{i \leq l} + \delta_{j \leq l} \cdot \delta_{i \leq k}$,

where $\delta_{x \leq y} := \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$

For a representation M in $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ of dimension vector $\underline{d_0}$ (or equivalently, for the corresponding 2-nilpotent matrix A), consider the corresponding oriented link pattern. Define p_k^M as the number of vertices to the left of k which are not incident with an arrow, plus the number of arrows whose target vertex is to the left of k. Define $q_{k,l}^M$ as p_l^M plus the number of arrows whose source vertex lies to the left of l and whose target vertex lies to the left of k.

Theorem 4.3 *We have $M \leq_{\deg} M'$ (or equivalently, $\mathcal{O}_{A'} \subset \overline{\mathcal{O}_A}$ in the notation above) if and only if $p_k^M \leq p_k^{M'}$ and $q_{k,l}^M \leq q_{k,l}^{M'}$ for all $k, l = 1, \ldots, n$.*

Proof Given two representations M and M', we write

$$M = \bigoplus_{i,j=1}^{n} \mathcal{U}_{i,j}^{m_{i,j}} \oplus \bigoplus_{i=1}^{n} \mathcal{V}_i^{n_i} \quad \text{and} \quad M' = \bigoplus_{i,j=1}^{n} \mathcal{U}_{i,j}^{m'_{i,j}} \oplus \bigoplus_{i=1}^{n} \mathcal{V}_i^{n'_i}.$$

Since we want to apply Theorem 4.1, we compute $[U, M]$ and $[U, M']$ for an indecomposable U and interpret these dimensions in terms of the oriented link patterns. The condition $[U, M] \leq [U, M']$ is equivalent to

$$\sum_{i,j=1}^{n} m_{i,j}[U, \mathcal{U}_{i,j}] + \sum_{i=1}^{n} n_i[U, \mathcal{V}_i] \leq \sum_{i,j=1}^{n} m'_{i,j}[U, \mathcal{U}_{i,j}] + \sum_{i=1}^{n} n'_i[U, \mathcal{V}_i].$$

Using the dimensions of homomorphism spaces between indecomposable representations provided by the previous lemma, we calculate

$$p_k^M = \sum_{i \leq k; j} m_{i,j} + \sum_{i \leq k} n_i$$

and

$$q_{k,l}^M = p_l^M + \sum_{i \leq k; j \leq l} m_{i,j},$$

and the condition $[U, M] \leq [U, M']$ is equivalent to the conditions $p_k^M \leq p_k^{M'}$ and $q_{k,l}^M \leq q_{k,l}^{M'}$ for all $k, l = 1, \ldots, n$. □

4.2 Minimal Degenerations

As a next step, we develop a combinatorial method to produce all degenerations of a given representation M in $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ of dimension vector $\underline{d_0}$ out of its oriented link pattern. It is sufficient to construct all minimal degenerations, that is, degenerations $M <_{\mathrm{deg}} M'$ such that if $M \leq_{\mathrm{deg}} L \leq_{\mathrm{deg}} M'$, then $M \cong L$ or $M' \cong L$. Minimal degenerations are denoted by $M <_{\mathrm{mdeg}} M'$.

In [11], Zwara describes all minimal degenerations; the result is stated here in a generality sufficient for our purposes. Denote by $\leq_{\mathrm{ext}}$ the transitive closure of the relation on representations given by $M \leq M'$ if there exists a short exact sequence $0 \to M'_1 \to M \to M'_2 \to 0$ such that $M' \cong M'_1 \oplus M'_2$.

Theorem 4.4 *Let M and M' be representations in* $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$.
If $M <_{\mathrm{mdeg}} M'$, then one of the following holds:

(1) $M <_{\mathrm{ext}} M'$.
(2) *There are representations W, $\widetilde{M}$, $\widetilde{M}'$ in* $\mathrm{rep}_k^{\mathrm{inj}}(\mathcal{Q}, I)$ *such that*

(a) $M \cong W \oplus \widetilde{M}$,
(b) $M' \cong W \oplus \widetilde{M'}$,
(c) $\widetilde{M} <_{\text{mdeg}} \widetilde{M'}$,
(d) $\widetilde{M'}$ *is indecomposable*.

Combining this theorem with the technique of [3, Theorem 4], we obtain a characterization of minimal disjoint degenerations, that is, minimal degenerations $M <_{\text{mdeg}} M'$ such that M and M' do not share a common direct summand:

Corollary 4.5 *Let $M <_{\text{mdeg}} M'$ be a minimal disjoint degeneration as before. Then either M' is indecomposable or $M' \cong U \oplus V$, where U and V are indecomposables and there exists an exact sequence $0 \to U \to M \to V \to 0$, or $0 \to V \to M \to U \to 0$.*

Proof By Theorem 4.4, either M' is indecomposable, or $M <_{\text{ext}} M'$. In the second case, due to the minimality of the degeneration, there exists a decomposition $M' \cong U \oplus V$ and an exact sequence $E : 0 \to U \to M \to V \to 0$. Assume that U' is a proper indecomposable direct summand of U. We can follow the argument of the proof of [3, Theorem 4] and consider the pushout E by the retraction $r : U \twoheadrightarrow U'$, yielding an exact sequence $0 \to U' \to X \to V \to 0$. We obtain

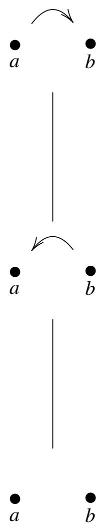
$$M \leq_{\text{deg}} K \oplus X <_{\text{deg}} K \oplus U' \oplus V = M',$$

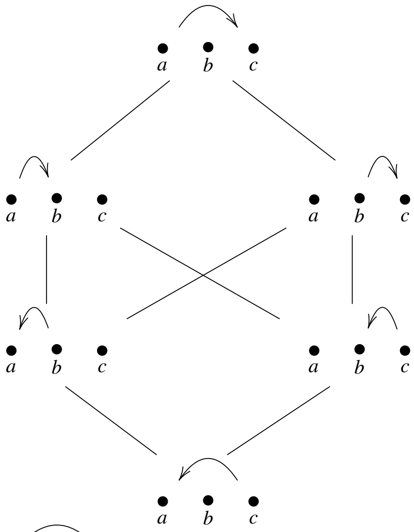
where K denotes the kernel of r. Thus, $M \cong K \oplus X$ due to the minimality of the degeneration, and K is a direct summand of both M and M', a contradiction.

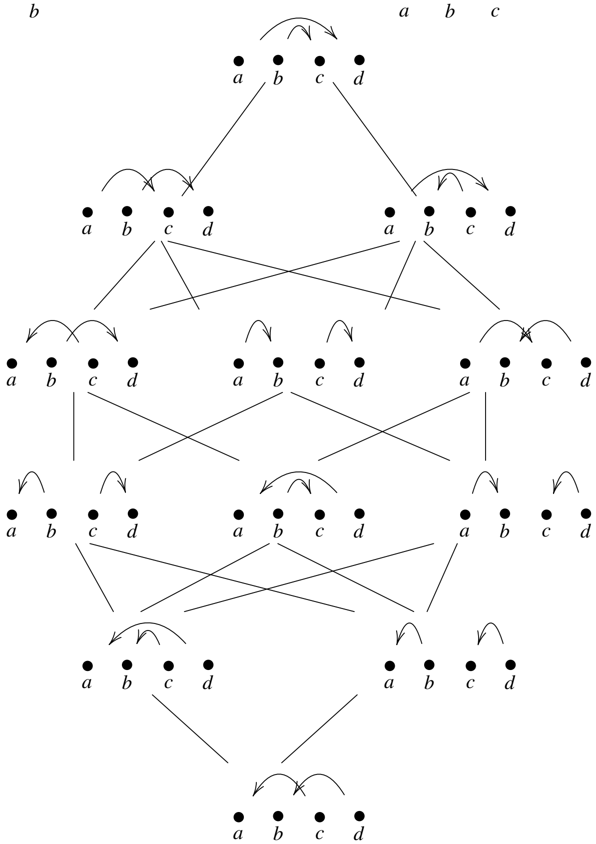
An analogous argument shows that V is indecomposable. □

Thus we see that all minimal degenerations are of the form $W \oplus M <_{\text{mdeg}} W \oplus M'$, where M and M' are as in the corollary, and thus M' involves at most two indecomposable direct summands. Translating this to the language of oriented link patterns using Theorem 3.4, we have "localized" the problem to the consideration of at most four vertices of an oriented link pattern. In this local case, we can apply Theorem 4.3 and easily work out all minimal degenerations.

Theorem 4.6 *Every minimal degeneration is of the form given in one of the following diagrams showing parts of the degeneration posets in terms of oriented link patterns. We assume that $a < b$ (resp. $a < b < c$, resp. $a < b < c < d$) are vertices of an oriented link pattern and only indicate the changes to the arrows incident with one of these vertices; all other arrows are left unchanged.*







Remark Note that, although every minimal degeneration is of the form $W \oplus M <_{\mathrm{mdeg}} W \oplus M'$ as above, the choice of W is not arbitrary, that is, addition of an arbitrary W might lead to a non-minimal degeneration. The precise conditions on W necessary for this degeneration to be minimal will be described in [4]; as a consequence, it will be shown in [4] that all minimal degenerations are of codimension 1.

We have thus obtained a constructive way of describing an orbit closure $\overline{\mathcal{O}_A}$ of a 2-nilpotent matrix A in terms of its corresponding oriented link pattern: by repeated application of the local changes to the arrows as in the theorem, we produce a list of all link patterns corresponding to matrices $A' \in N_n^{(2)}$ such that $\mathcal{O}_{A'} \subset \overline{\mathcal{O}_A}$ (although this list will contain repetitions due to non-minimal degenerations).

5 B_n-Orbits in Arbitrary Nilpotent Matrices

In this section, we consider the conjugation action of B_n on arbitrary nilpotent matrices (for the analogous problem of the action of B_n on $\mathfrak{n}_n$, see [6]).

We start with a general remark explaining our strategy: suppose we are given the action of an algebraic group G on a variety X for which a complete classification of the orbits cannot be obtained. Then there are several strategies for a partial classification: we can restrict attention to a natural closed subvariety $X_1 \subset X$ on which G acts with finitely many orbits, which can then be classified explicitly (by some discrete, combinatorial labeling). Or we can classify the "generic" orbits forming a natural open subset $U \subset X$ (typically by continuous parameters). Moreover, we can try to describe a quotient of X by G by constructing G-invariant functions on X.

Whereas the first strategy for the action of B_n on $\mathcal{N}_n$ is pursued in the previous sections, we concentrate on a classification and description of generic orbits in Sect. 5.1. Finally, we construct B_n-(semi-)invariant functions on $\mathcal{N}_n$ in Sect. 5.2.

The starting point is the following observation (see [5]):

Example Consider the action of B_3 on $\mathcal{N}_3$ via conjugation. Then the matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda & 1 & 0 \end{bmatrix} \quad \text{for } \lambda \in k$$

are pairwise non-conjugate. Furthermore, on the open set $U \subset \mathcal{N}_3$ of nilpotent matrices

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

where $g \neq 0$ or $dh \neq eg$, the map $U \to \mathbf{P}^1$ given by $A \mapsto (g : dh - eg)$ is surjective and B_3-invariant.

We generalize some aspects of this example to arbitrary n.

5.1 Generic Normal Form

It is appropriate to reformulate the problem as follows: we consider the action of $\mathrm{GL}(V)$ on pairs (F_*, φ) consisting of a complete flag $0 = F_0 \subset F_1 \subset \cdots \subset F_n = V$ and a nilpotent operator $\varphi \in \mathrm{End}(V)$ of an n-dimensional k-vector space V. Then the orbits of this action are precisely the orbits of B_n in $\mathcal{N}_n$ since the variety of complete flags is isomorphic to the homogeneous space GL_n/B_n.

Theorem 5.1 *The following properties of a pair (F_*, φ) consisting of a complete flag and a nilpotent operator of an n-dimensional k-vector space V are equivalent*:

(1) $\dim \varphi^{n-k}(F_k) = k$ *for all* $k = 1, \dots, n-1$,
(2) $\dim(\varphi^{n-k}(F_k) + F_{n-k})/F_{n-k} = k$ *for all* $k = 1, \dots, n-1$, *or, equivalently, all induced maps* $\varphi^{n-k} : F_k \to V/F_{n-k}$ *are invertible*,
(3) *there exists a unique basis* $v_1, \dots, v_n$ *of* V *such that*
 (a) $F_k = \langle v_1, \dots, v_k \rangle$ *for all* $k = 0, \dots, n$,
 (b) $\varphi(v_k) = v_{k+1} \bmod \langle v_{k+2}, \dots, v_n \rangle$ *for all* $k = 1, \dots, n$.

Proof Obviously, the second property implies the first. We show that the third property implies the second one; so assume that there exists a basis $v_1, \dots, v_n$ with the properties (a) and (b). By an easy induction, we have

$$\varphi^k(v_l) = v_{k+l} \bmod \langle v_{k+l+1}, \dots, v_n \rangle$$

for all $k + l \leq n$, and $\varphi^k(v_l) = 0$ if $k + l > n$. We thus have

$$\varphi^{n-k}(F_k) = \langle \varphi^{n-k}(v_1), \dots, \varphi^{n-k}(v_k) \rangle = \langle v_{n-k+1}, \dots, v_n \rangle,$$

and the second property follows since $F_{n-k} = \langle v_1, \dots, v_{n-k} \rangle$.

Conversely, assume that $\dim \varphi^{n-k}(F_k) = k$ for all k. In particular, we have $\varphi^{n-k}(V) = \varphi^{n-k}(F_k)$, and thus $\dim \varphi^{n-k}(V) = k$ and $\dim \mathrm{Ker}(\varphi^{n-k}) = n - k$ for all k. We choose an arbitrary basis $w_1, \dots, w_n$ of V which is adapted to F_*, that is, such that $F_k = \langle w_1, \dots, w_k \rangle$ for all k. Then, for all k, the elements $\varphi^{n-k}(w_1), \dots, \varphi^{n-k}(w_k)$ generate the k-dimensional space $\varphi^{n-k}(F_k)$, and thus they form a basis of this space. We can thus write the element $\varphi^{n-1}(w_1) \in \varphi^{n-k}(F_k)$ uniquely as

$$\varphi^{n-1}(w_1) = \sum_{i=1}^{k} b_{k,i} \varphi^{n-k}(w_i),$$

and we define

$$v_k = \sum_{i=1}^{k} b_{k,i} w_i$$

for all k. Note that the elements v_k do not depend on the choice of basis elements $w_1, \dots, w_n$. We have $b_{k,k} \neq 0$: otherwise $\varphi^{n-1}(w_1) = \sum_{i<k} b_{k,i} \varphi^{n-k}(w_i)$, and application of φ yields $0 = \sum_{i<k} b_{k,i} \varphi^{n-(k-1)}(w_i)$ and thus $b_{k,i} = 0$ for all i by linear

independence of the elements $\varphi^{n-(k-1)}(w_i)$. Then $\varphi^{n-1}(w_1)=0$, a contradiction. Since the elements w_k form a basis and the $b_{k,k}$ are non-zero, the elements v_k form a basis, too, which is again adapted to F_*.

We have

$$\varphi^{n-k}(v_k)=\sum_{i=1}^{k} b_{k,i}\varphi^{n-k}(w_i)=\varphi^{n-1}(w_1)=v_n$$

by definition. For $k+l>n$, we thus have

$$\varphi^k(v_l)=\varphi^{k+l-n}\big(\varphi^{n-l}(v_l)\big)=\varphi^{k+l-n}\big(\varphi^{n-1}(w_1)\big)=0.$$

It follows that $v_{k+1},\dots,v_n$ belong to $\mathrm{Ker}(\varphi^{n-k})$, and thus they form a basis of this space for dimension reasons. It also follows that $\varphi^{n-k}(v_1),\dots,\varphi^{n-k}(v_k)$ form a basis of $\varphi^{n-k}(V)$.

Writing $\varphi^k(v_l)=\sum_{i=1}^n c_{k,l,i}v_i$, we apply φ^{n-k-l} and calculate

$$\begin{aligned}\varphi^{n-k-l}(v_{k+l})=v_n=\varphi^{n-l}(v_l)&=\sum_i c_{k,l,i}\varphi^{n-k-l}(v_i)\\&=c_{k,l,k+l}\varphi^{n-k-l}(v_{k+l})+\sum_{i<k+l} c_{k,l,i}\varphi^{n-k-l}(v_i),\end{aligned}$$

and thus $c_{k,l,k+l}=1$ and $c_{k,l,i}=0$ for all $i<k+l$ by linear independence of $\varphi^{n-k-l}(v_1),\dots,\varphi^{n-k-l}(v_{k+l})$. We thus have

$$\varphi^k(v_l)=v_{k+l}+\sum_{i>k+l} c_{k,l,i}v_i$$

for all $k+l\le n$, and, in particular,

$$\varphi(v_k)=v_{k+1}+\sum_{i>k+1} c_{1,k,i}v_i$$

for all k. The basis $v_1,\dots,v_n$ thus has the claimed properties. □

For $0\le a,b\le n$ and a matrix $A\in\mathcal{N}_n$, define $A_{(a,b)}$ as the submatrix formed by the last a rows and the first b columns of A.

Corollary 5.2 *The following conditions on a matrix $A\in\mathcal{N}_n$ are equivalent*:

(1) *for $k=1,\dots,n-1$, the first k columns of A^{n-k} are linearly independent*,
(2) *for $k=1,\dots,n-1$, the minor* $\det((A^{n-k})_{(k,k)})$ *is non-zero*,
(3) *A is B_n-conjugate to a unique matrix H such that $H_{i,j}=0$ for $i\le j$ and $H_{i+1,i}=1$ for all $i=1,\dots,n-1$.*

Proof We apply the previous theorem to the vector space $V=\mathbf{k}^n$ with coordinate basis $e_1,\dots,e_n$, the standard flag defined by $F_k=\langle e_1,\dots,e_k\rangle$ and the endomorphism φ given by multiplication by A. The first property of the theorem immediately translates into linear independence of column vectors, whereas the second property translates to the non-vanishing of minors. The basis $v_1, \ldots, v_n$ of the theorem yields an upper-triangular base change matrix, and representing A with respect to this basis yields the desired B_n-conjugate H. □

The conditions of Corollary 5.2 define an open subset of $\mathcal{N}_n$; we have thus found a generic normal form for nilpotent matrices up to B_n-conjugacy.

5.2 Semiinvariants

We construct a class of determinantal B_n-semiinvariants on $\mathcal{N}_n$, that is, regular functions D on $\mathcal{N}_n$ such that $D(gAg^{-1}) = \chi(g)D(A)$ for all $g \in B_n$ and $A \in \mathcal{N}_n$; here χ is a character on B_n called the weight of D. For $i = 1, \ldots, n$, we denote by $\omega_i : B_n \to \mathbf{G}_m$ the character defined by $\omega_i(g) = g_{i,i}$; the ω_i form a basis for the group of characters of B_n.

Fix non-negative integers $a_1, \ldots, a_s, b_1, \ldots, b_t$ such that $a_1 + \cdots + a_s = b_1 + \cdots + b_t =: k \leq n$. Moreover, fix polynomials $P_{i,j}(x) \in k[x]$ for $i = 1, \ldots, s$ and $j = 1, \ldots, t$, and denote the datum $((a_i)_i, (b_j)_j, (P_{i,j})_{i,j})$ by P. For all such i and j, consider the $a_i \times b_j$-submatrices $P_{i,j}(A)_{(a_i,b_j)}$ as defined in the previous section, and form the block matrix $A^P = (P_{i,j}(A)_{(a_i,b_j)})_{i,j}$; this is a $k \times k$-matrix.

Proposition 5.3 *For every datum P as above, the function associating to a matrix $A \in \mathcal{N}_n$ the determinant $\det(A^P)$ defines a B_n-semiinvariant regular function D^P of weight $\sum_i(\omega_{n-a_i+1} + \cdots + \omega_n) - \sum_j(\omega_1 + \cdots + \omega_{b_j})$ on $\mathcal{N}_n$.*

Proof For $g \in B_n$ and $1 \leq a, b \leq n$, denote by $g_{(\geq a)} \in B_a$ (resp. by $g_{(\leq b)} \in B_b$) the submatrix formed by the last a rows and columns (resp. by the first b rows and columns) of g. With these definitions, it follows immediately that

$$\left(gAg^{-1}\right)_{(a,b)} = g_{(\geq a)}A_{(a,b)}g_{(\leq b)}^{-1}.$$

This yields the following equalities of block matrices:

$$\begin{aligned}\left(gAg^{-1}\right)^P &= \left(P_{i,j}\left(gAg^{-1}\right)_{(a_i,b_j)}\right)_{i,j} = \left(\left(gP_{i,j}(A)g^{-1}\right)_{(a_i,b_j)}\right)_{i,j} \\ &= \left(g_{(\geq a_i)}P_{i,j}(A)_{(a_i,b_j)}{g_{(\leq b_j)}}^{-1}\right)_{i,j} = \left(\delta_{i,j}g_{(\geq a_i)}\right)_{i,j}A^P\left(\delta_{i,j}{g_{(\leq b_j)}}^{-1}\right)_{i,j},\end{aligned}$$

and thus

$$D^P\left(gAg^{-1}\right) = \det\left(\left(gAg^{-1}\right)^P\right) = \prod_i \det(g_{(\geq a_i)}) \prod_j \det(g_{(\leq b_j)})^{-1} D^P(A).$$

□

With the aid of these semiinvariants, we can see that the entries of the normal form H associated to a matrix A fulfilling the conditions of Corollary 5.2 depend polynomially on A, by describing them as the value of a special semiinvariant D^P:

Lemma 5.4 *For i and j such that $1 \leq j \leq n-2$ and $j+2 \leq i \leq n$, consider the datum P as above defined by $a_1 = j-1$, $a_2 = n+1-i$, $b_1 = j$, $b_2 = n-i$, $P_{1,1}(x) = x^{n-j+1}$, $P_{1,2}(x) = 0$, $P_{2,1}(x) = x$, $P_{2,2}(x) = x^i$. Then, for a matrix H in the form of Corollary 5.2, we have $D^P(H) = H_{i,j}$.*

Proof By a direct calculation, the matrix H^P consists of the blocks

$$\left(H^P\right)_{1,1} = \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \dots & 1 & 0 \end{pmatrix}, \qquad \left(H^P\right)_{1,2} = 0,$$

$$\left(H^P\right)_{2,1} = H_{(n-i+1,j)}, \qquad \left(H^P\right)_{2,2} = \begin{pmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ * & \dots & 1 \end{pmatrix}.$$

Thus, the matrix H^P is lower triangular, all diagonal entries being 1 except the (j,j)-entry, which equals $H_{i,j}$. □

It seems likely that the semiinvariants D^P generate the ring of all semiinvariants at least for a certain cone of weights. The generic normal form of Corollary 5.2 allows us to find identities between the D^P by evaluation on matrices H in normal form.

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Weyl Denominator Identity for Finite-Dimensional Lie Superalgebras

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Abstract Weyl denominator identity for the basic simple Lie superalgebras was formulated by Kac and Wakimoto and was proven by them for the defect one case. In this paper we prove the identity for the rest of the cases.

Keywords Lie superalgebra

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1 Introduction

The basic simple Lie superalgebras are finite-dimensional simple Lie superalgebras, which have a reductive even part and admit an even non-degenerate invariant bilinear form. These algebras were classified by Kac [K1], and the list (excluding Lie algebra case) consists of four series: $A(m,n)$, $B(m,n)$, $C(m)$, $D(m,n)$ and the exceptional algebras $D(2,1,a)$, $F(4)$, $G(3)$.

Let $\mathfrak{g}$ be a basic simple Lie superalgebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and let $\Delta_+ = \Delta_{+,0} \sqcup \Delta_{+,1}$ be the corresponding set of positive roots. The Weyl denominator associated to the above data is

$$R := \frac{\prod_{\alpha\in\Delta_{+,0}}(1-e^{-\alpha})}{\prod_{\alpha\in\Delta_{+,1}}(1+e^{-\alpha})}.$$

If $\mathfrak{g}$ is a finite-dimensional simple Lie algebra (i.e. $\Delta_1 = \emptyset$), then the Weyl denominator is given by the Weyl denominator identity

$$Re^{\rho} = \sum_{w\in W} \text{sgn}(w) e^{w\rho},$$

where ρ is the half-sum of the positive roots, W is the Weyl group, i.e. the subgroup of $GL(\mathfrak{h}^*)$ generated by the reflections with respect to the roots, and $\text{sgn}(w) \in \{\pm 1\}$

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is the sign of $w \in W$. This identity may be viewed as the character of trivial representation of the corresponding Lie algebra.

The Weyl denominator identities for superalgebras were formulated and partially proven (for $A(m-1, n-1)$, $B(m, n)$, $D(m, n)$ with $\min(m, n) = 1$ and for $C(n)$, $D(2, 1, a)$, $F(4)$, $G(3)$) by Kac and Wakimoto [KW]. In order to state the Weyl denominator identity for basic simple Lie superalgebras, we need the following notation. Let $\Delta^{\#}$ be the "largest" component of Δ_0, see 3.2 for the definition. Let W be the Weyl group of $\mathfrak{g}_0$, i.e. the subgroup of $GL(\mathfrak{h}^*)$ generated by the reflections with respect to the even roots Δ_0, and let $\text{sgn}(w)$ be the sign of w. One has $W = W_1 \times W_2$, where $W^{\#}$ is the Weyl group of root system $\Delta^{\#}$, i.e. the subgroup of W generated by the reflections with respect to the roots from $\Delta^{\#}$. Set $\rho_0 := \sum_{\alpha \in \Delta_{+,0}} \alpha/2$, $\rho_1 := \sum_{\alpha \in \Delta_{+,1}} \alpha/2$, $\rho := \rho_0 - \rho_1$. A subset Π of Δ_+ is called a *set of simple roots* if the elements of Π are linearly independent and $\Delta_+ \subset \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0}\alpha$. A subset S of Δ is called *maximal isotropic* if the elements of S form a basis of a maximal isotropic space in $\mathbb{R}\Delta$. By [KW], Δ contains a maximal isotropic subset, and each maximal isotropic subset is a subset of a set of simple roots (for a certain triangular decomposition). Fix a maximal isotropic subset $S \subset \Delta$ and choose a set of simple roots Π containing S. Let R be the Weyl denominator for the corresponding triangular decomposition. The following Weyl denominator identity was suggested by Kac and Wakimoto [KW]:

$$Re^{\rho} = \sum_{w \in W^{\#}} \text{sgn}(w) w \left(\frac{e^{\rho}}{\prod_{\beta \in S}(1 + e^{-\beta})} \right). \tag{1}$$

If S is empty (i.e. $(-,-)$ is a positive/negative definite), the denominator identity takes the form $Re^{\rho} = \sum_{w \in W} \text{sgn}(w) e^{w\rho}$. In this case, either Δ_1 is empty (i.e. $\mathfrak{g}$ is a Lie algebra), or $\mathfrak{g} = \mathfrak{osp}(1, 2l)$ (type $B(0, l)$). The Weyl denominator identity for the case $\mathfrak{osp}(1, 2l)$ was proven in [K2]; for the case where S has the cardinality one, the identity was proven in [KW].

The Weyl denominator identity for root system of Lie (super)algebra $\mathfrak{g}$ can be again naturally interpreted as the character of one-dimensional representation of $\mathfrak{g}$. The proofs in the above-mentioned cases [K2, KW] are based on a analysis of the highest weights of irreducible subquotients of the Verma module $M(0)$ over $\mathfrak{g}$. In this paper we give a proof of the Weyl denominator identity (1) for the case where S has the cardinality greater than one. A similar proof works for the case where the cardinality of S is one. Unfortunately, our proof does not use representation theory, but requires an analysis of the roots systems. The proof is based on a case-by-case verification of the following facts:

(i) the monomials appearing in the right-hand side of (1) are of the form $e^{\rho-\nu}$, where $\nu \in Q^+ := \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0}\alpha$, and the coefficient of e^{ρ} is one;
(ii) the right-hand side of (1) is W-skew-invariant (i.e. $w \in W$ acts by the multiplication by $\text{sgn}(w)$).

Taking into account that for any $\lambda \in V$, the stabilizer of λ in W is either trivial or contains a reflection, and that if the stabilizer is trivial and $W\lambda \subset (\rho_0 - \sum_{\alpha \in \Delta_+} \mathbb{Z}_{\geq 0}\alpha)$, then $\lambda = \rho_0$, we easily deduce identity (1) from (i) and (ii).

I. Musson informed us that he has an unpublished proof of the Weyl denominator identity for basic simple Lie superalgebras.

The Weyl denominator identity for the affinization of a simple finite-dimensional Lie superalgebra with non-zero Killing form was also formulated by Kac and Wakimoto [KW] and was proven for the defect one case. We prove this identity in [G].

2 The Algebra $\mathcal{R}$

In this section we introduce the algebra $\mathcal{R}$. Since the main technical difficulty in our proof of the denominator identity comes from the existence of different triangular decompositions, we illustrate how the proof works when there is only one triangular decomposition (see 2.3, 2.4).

2.1 Notation Set $V=\mathfrak{h}^*_{\mathbb{R}}$ (so $V=\text{span}\,\Delta$ for $\mathfrak{g}\neq A(m,n)$ and $V=\mathbb{R}\,\text{span}\,\Delta\oplus\mathbb{R}$ for $\mathfrak{g}=A(m,n)$). Let $Q\subset V$ be the root lattice, and let Q^+ be the positive part of root lattice $Q^+:=\sum_{\alpha\in\Pi}\mathbb{Z}_{\geq 0}\alpha$. We introduce the following partial order on V:

$$\mu\leq\nu \quad \text{if } (\nu-\mu)\in\sum_{\alpha\in\Delta_+}\mathbb{R}_{\geq 0}\alpha.$$

Introduce the height function $\text{ht}:Q^+\to\mathbb{Z}_{\geq 0}$ by

$$\text{ht}\left(\sum_{\alpha\in\Pi}m_\alpha\alpha\right):=\sum_{\alpha\in\Pi}m_\alpha.$$

We use the following notation: for $X\subset\mathbb{R}$ and $Y\subset V$, we set $XY:=\{xy|x\in X, y\in Y\}$; for instance, $Q^+:=\mathbb{Z}_{\geq 0}\Pi$.

Note that Δ_0 is the root system of the reductive Lie algebra $\mathfrak{g}_0$. In particular, all isotropic roots are odd. Both sets Δ_0,Δ_1 are W-stable: $W\Delta_i=\Delta_i$.

2.2 The Algebra $\mathcal{R}$ Denote by $\mathbb{Q}[e^\nu,\ \nu\in V]$ the algebra of polynomials in e^ν, $\nu\in V$. Let $\mathcal{R}$ be the algebra of rational functions of the form

$$\frac{X}{\prod_{\alpha\in\Delta_+}(1+a_\alpha e^{-\alpha})^{m_\alpha}},$$

where $X\in\mathbb{Q}[e^\nu,\ \nu\in V]$, and $a_\alpha\in\mathbb{Q}$, $m_\alpha\in\mathbb{Z}_{\geq 0}$. Clearly, $\mathcal{R}$ contains the rational functions of the form $\frac{X}{\prod_{\alpha\in\Delta}(1+a_\alpha e^{-\alpha})^{m_\alpha}}$ with X, a_α, m_α as above. The group W acts on $\mathcal{R}$ by the automorphism mapping e^ν to $e^{w\nu}$. We say that $P\in\mathcal{R}$ is W-*invariant* (resp., W-*skew-invariant*) if $wP=P$ (resp., $wP=\text{sgn}(w)P$) for every $w\in W$.

2.2.1 For a sum $Y := \sum b_\mu e^\mu$, $b_\mu \in \mathbb{Q}$, introduce the *support* of Y by the formula

$$\operatorname{supp}(Y) := \{\mu \mid b_\mu \neq 0\}.$$

Any element of $\mathcal{R}$ can be uniquely expanded in the form

$$\frac{\sum_{i=1}^m a_i e^{\nu_i}}{\prod_{\alpha\in\Delta_+}(1+a_\alpha e^{-\alpha})^{m_\alpha}} = \sum_{i=1}^s \sum_{\mu\in Q^+} b_\mu e^{\nu_i-\mu}, \quad b_\mu \in \mathbb{Q}.$$

For $Y \in \mathcal{R}$, denote by $\operatorname{supp}(Y)$ the support of its expansion; by the above, $\operatorname{supp}(Y)$ lies in a finite union of cones of the form $\nu - Q^+$.

2.2.2 We call $\lambda \in V$ *regular* if $\operatorname{Stab}_W \lambda = \{\mathrm{id}\}$; we call the orbit $W\lambda$ regular if λ is regular (so the orbit consists of regular points).

It is well known that for $\lambda \in V$, the stabilizer $\operatorname{Stab}_W \lambda$ is either trivial or contains a reflection (see Lemma 5.1.1(ii)). Therefore the stabilizer of a non-regular point $\lambda \in V$ contains a reflection. As a result, the space of W-skew-invariant elements of $\mathbb{Q}[e^\nu,\ \nu \in V]$ is spanned by $\sum_{w\in W} \operatorname{sgn}(w) e^{w\lambda}$, where $\lambda \in V$ is regular. In particular, the support of a W-skew-invariant element of $\mathbb{Q}[e^\nu,\ \nu \in V]$ is a union of regular W-orbits.

2.3 Lie Algebra Case The denominator identity for Lie algebra is

$$Re^\rho = \prod_{\alpha\in\Delta_+}\left(1-e^{-\alpha}\right)e^\rho = \sum_{w\in W} \operatorname{sgn}(w) e^{w\rho}.$$

There are several proofs of this identity. The proof we are going to generalize is the following. Observe that $Re^\rho \in \mathbb{Q}[e^\nu,\ \nu \in V]$ and $\operatorname{supp}(Re^\rho) \subset (\rho - Q^+)$. Moreover, $Re^\rho = \prod_{\alpha\in\Delta_+}(e^{\alpha/2} - e^{-\alpha/2})$ is W-skew-invariant, so $\operatorname{supp}(Re^\rho)$ is a union of regular orbits lying in $(\rho - Q^+)$. However, $W\rho$ is the only regular orbit lying entirely in $(\rho - Q^+)$, see Lemma 5.1.1(iii). Hence, Re^ρ is proportional to $\sum_{w\in W} \operatorname{sgn}(w) e^{w\rho}$. Since the coefficient of e^ρ in the expression Re^ρ is 1, the coefficient of proportionality is 1.

2.4 Case $Q(n)$ For the case $\mathfrak{g} := Q(n)$, we have $\mathfrak{g}_0 = \mathfrak{gl}(n)$, $W = S_n$, $\Delta_{0+} = \Delta_{+,1} = \{\varepsilon_i - \varepsilon_j\}_{1\le i<j\le n}$, and so $\rho_0 = \rho_1$, $\rho = 0$. The Weyl denominator is $R = \prod_{\alpha\in\Delta_{+,0}} \frac{1-e^{-\alpha}}{1+e^{-\alpha}}$. For each $S \subset \Delta_{+,1}$, define

$$A(S) := \{w \in S_n \mid wS \subset \Delta_{+,0}\}, \qquad a(S) := \sum_{w\in A(S)} \operatorname{sgn}(w).$$

2.4.1. Proposition *For each $S \subset \Delta_{+,1}$, we have*

$$a(S)R = \sum_{w\in S_n} \operatorname{sgn}(w) \frac{1}{\prod_{\beta\in S}(1+e^{-w\beta})}.$$

Proof Observe that the support of both sides of the formula lies in $-Q^+$ and that the coefficients of $1 = e^0$ in both sides are equal to $a(S)$. Multiplying both sides of the formula by the W-invariant expression $\prod_{\alpha\in\Delta_{+,1}}(1+e^{-\alpha})e^{\rho_1} = \prod_{\alpha\in\Delta_{+,1}}(e^{\alpha/2} + e^{-\alpha/2})$, we obtain

$$Y := a(S)\prod_{\alpha\in\Delta_{+,0}}\left(1-e^{-\alpha}\right)e^{\rho_0} - \prod_{\alpha\in\Delta_{+,1}}\left(1+e^{-\alpha}\right)e^{\rho_1}\sum_{w\in S_n}\operatorname{sgn}(w)\frac{1}{\prod_{\beta\in S}(1+e^{-w\beta})}$$

$$= a(S)\prod_{\alpha\in\Delta_{+,0}}\left(1-e^{-\alpha}\right)e^{\rho_0} - \sum_{w\in S_n}\operatorname{sgn}(w)w\Bigg(\prod_{\alpha\in\Delta_{+,0}\setminus S}\left(1+e^{-\alpha}\right)e^{\rho_0}\Bigg).$$

Since $\prod_{\alpha\in\Delta_{+,0}}(1-e^{-\alpha})e^{\rho_0} = \prod_{\alpha\in\Delta_{+,0}}(e^{\alpha/2} - e^{-\alpha/2})$ is W-skew-invariant, Y is also W-skew-invariant. Clearly, $Y \in \mathbb{Q}[e^{\nu},\ \nu \in V]$, so, by 2.2.2, $\operatorname{supp}(Y)$ is a union of regular W-orbits. By the above, $\operatorname{supp}(Y) \subset (\rho_0 - Q^+)\setminus\{\rho_0\}$. However, by Lemma 5.1.1(iii), any regular W-orbit intersects $\rho_0 + Q^+$. Hence $Y = 0$, as required. □

2.4.2 In order to obtain a formula for the Weyl denominator R, we choose S such that $a(S) \neq 0$. Taking $S = \{\varepsilon_1 - \varepsilon_n, \varepsilon_2 - \varepsilon_{n-1}, \ldots, \varepsilon_{[\frac{n}{2}]} - \varepsilon_{n+1-[\frac{n}{2}]}\}$ and using Lemma 2.4.3, we obtain the following formula:

$$R = \frac{1}{[n/2]!}\sum_{w\in S_n}\operatorname{sgn}(w)\frac{1}{\prod_{\beta\in S}(1+e^{-w\beta})},$$

which appears in [KW] (7.1) (up to a constant factor).

Note that such S has a minimal cardinality: if the cardinality of S is less than $[\frac{n}{2}]$, then $a(S) = 0$. Indeed, if the cardinality of S is less than $[\frac{n}{2}]$, then there is a root $\varepsilon_i - \varepsilon_j$ which does not belong to the span of S, and thus $s_{\varepsilon_i-\varepsilon_j}W(S) = W(S)$. Since $\operatorname{sgn}(w) + \operatorname{sgn}(s_{\varepsilon_i-\varepsilon_j}w) = 0$, this forces $a(S) = 0$.

2.4.3. Lemma *Set* $A := \{\sigma \in S_n \mid \forall i \le \frac{n}{2},\ \sigma(i) > \sigma(n+1-i)\}$. *Then*

$$\sum_{\sigma\in A}\operatorname{sgn}(\sigma) = [n/2]!$$

Proof For each $\sigma \in A$, let $P(\sigma)$ be the set of pairs $\{(\sigma(1),\sigma(n)); (\sigma(2),\sigma(n-1)); \ldots\}$, that is $P(\sigma) := \{(\sigma(j),\sigma(n+1-j))\}_{j=1}^{[\frac{n}{2}]}$. Let $B := \{\sigma \in A \mid P(\sigma) = \{(j, j+1)\}_{j=1}^{[\frac{n}{2}]}\}$. Define an involution f on the set $A\setminus B$ as follows: for $\sigma \in A\setminus B$, set $f(\sigma) := (i, i+1)\circ\sigma$, where i is minimal such $(i, i+1) \notin P(\sigma)$. Since $\operatorname{sgn}(\sigma) + \operatorname{sgn}(f(\sigma)) = 0$, we get $\sum_{\sigma\in A\setminus B}\operatorname{sgn}(\sigma) = 0$. We readily see that B has $[n/2]!$ elements and that $\operatorname{sgn}(\sigma) = 1$ for each $\sigma \in B$. Hence $\sum_{\sigma\in A}\operatorname{sgn}(\sigma) = [n/2]!$, as required. □

3 Notation

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a basic Lie superalgebra with a fixed triangular decomposition of the even part: $\mathfrak{g}_0 = \mathfrak{n}_{-,0} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+,0}$. For $A(m-1, n-1)$-type, we put $\mathfrak{g} = \mathfrak{gl}(m|n)$ (we readily see that the denominator identities for $\mathfrak{gl}(m|n)$ imply the denominator identities for $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(n|n)$). Let Δ_0 (resp., Δ_1) be the set of even (resp., odd) roots of $\mathfrak{g}$. As before, set $V = \mathfrak{h}^*_{\mathbb{R}}$. Denote by $(-,-)$ a non-degenerate symmetric bilinear form on V, induced by a non-degenerate invariant bilinear form on $\mathfrak{g}$. Retain the notation of Sect. 1 and define the Weyl denominator.

The dimension of a maximal isotropic space in $V = \mathfrak{h}^*_{\mathbb{R}}$ is called the *defect* of $\mathfrak{g}$. If $\mathfrak{g}$ is a Lie algebra or $\mathfrak{g} = \mathfrak{osp}(1, 2l)$ (type $B(0, l)$), then the defect of $\mathfrak{g}$ is zero; the defect of $A(m-1, n-1)$, $B(m, n)$, $D(m, n)$ is equal to $\min(m, n)$; for $C(n)$ and the exceptional Lie superalgebras, the defect is equal to one. Notice that the cardinality of a maximal isotropic set S is equal to defect of $\mathfrak{g}$.

3.1 Admissible Pairs The set of positive even roots $\Delta_{+,0}$ is determined by the triangular decomposition $\mathfrak{g}_0 = \mathfrak{n}_{-,0} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+,0}$ (i.e. $\Delta_{+,0}$ is the set of weights of $\mathfrak{n}_{+,0}$). Recall that for each maximal isotropic set S, there exists a set of simple roots containing S. We call a pair (S, Π) *admissible* if $S \subset \Pi$ is a maximal isotropic set of roots and Π is a set of simple roots such that the corresponding set of positive even roots coincides with $\Delta_{+,0}$:

$$(S, \Pi) \text{ is admissible if } S \subset \Pi \ \& \ \Delta_+(\Pi) \cap \Delta_0 = \Delta_{+,0}.$$

For a fixed set of simple roots Π, we retain the notation of 2.1 and 2.2.

3.2 The Set $\Delta^\#$ Let Δ_1, Δ_2 be two finite irreducible root systems. We say that Δ_1 is "larger" than Δ_2 if either the rank of Δ_1 is greater than the rank of Δ_2, or the ranks are equal and $\Delta_1 \subset \Delta_2$.

If the defect of $\mathfrak{g}$ is greater than one, then the root system Δ_0 is a disjoint union of two irreducible root systems. We denote by $\Delta^\#$ the irreducible component, which is not the smallest one, i.e. $\Delta_0 = \Delta^\# \sqcup \Delta_2$, where $\Delta^\#$ is not smaller than Δ_2, see the following table:

Δ	$A(m-1,n-1)$		$B(m,n)$			$D(m,n)$	
	$m>n$	$m \le n$	$m>n$	$m<n$	$m=n$	$m>n$	$m \le n$
$\Delta^\#$	A_{m-1}	A_{n-1}	B_m	C_n	B_m or C_m	D_m	C_n

The notion of $\Delta^\#$ in [KW] coincides with the above one, except for the case $B(m, m)$, where we allow both choices B_m and C_m, whereas in [KW], $\Delta^\#$ is of the type C_m.

Notice that $\mathfrak{g}_0 = \mathfrak{s}_1 \times \mathfrak{s}_2$, where $\mathfrak{s}_1, \mathfrak{s}_2$ are reductive Lie algebras, and $\Delta^\#$, $\Delta_0 \setminus \Delta^\#$ are roots systems of $\mathfrak{s}_1, \mathfrak{s}_2$ respectively. We normalize $(-,-)$ in such a way that $\Delta^\# := \{\alpha \in \Delta_0 | \ (\alpha, \alpha) > 0\}$. Then $\Delta_0 \setminus \Delta^\# = \{\alpha \in \Delta_0 | \ (\alpha, \alpha) < 0\}$.

4 Outline of the Proof

4.1 Let $\mathfrak{g}$ be one of the Lie superalgebras $A(m-1,n-1)$, $B(m,n)$, $D(m,n)$, $m,n>0$.

4.2 Expansion of the Right-Hand Side of (1) Let (S,Π) be an admissible pair. Set

$$X := \sum_{w\in W^{\#}} \mathrm{sgn}(w) w\left(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\right) \tag{2}$$

and rewrite the denominator identity (1) as $Re^{\rho}=X$.

Expanding X, we obtain

$$X = \sum_{w\in W^{\#}} \sum_{\mu\in\mathbb{Z}_{\geq 0}S} \mathrm{sgn}(w)(-1)^{\mathrm{ht}\,\mu} e^{\varphi(w)-|w|\mu+w\rho}, \tag{3}$$

where

$$\varphi(w) := \sum_{\beta\in S: w\beta<0} w\beta \in -Q^{+},$$

and $|w|$ is a linear map $\mathbb{Z}_{\geq 0}S \to Q^{+}$ defined on $\beta\in S$ by the formula

$$|w|\beta = \begin{cases} w\beta & \text{for } w\beta>0,\\ -w\beta & \text{for } w\beta<0.\end{cases}$$

4.3 Main Steps The proof has the following steps:

(i) We introduce certain operations on the admissible pairs (S,Π) and show that these operations preserve the expressions X and Re^{ρ}. Consider the equivalence relation on the set of admissible pairs (S,Π) generated by these operations. We will show that there are two equivalence classes for $D(m,n), m>n$ and one equivalence class for other cases.

(ii) We check that $\mathrm{supp}(X)\subset(\rho-Q^{+})$ and that the coefficient of e^{ρ} in X is 1 for a certain choice of (S,Π) (for $D(m,n)$, $m>n$, we check this for (S,Π) and (S',Π), which are representatives of the equivalence classes).

(iii) We show that X is W-skew-invariant for a certain choice of (S,Π) (for $D(m,n)$, $m>n$, we show this for (S,Π) and (S',Π), which are representatives of the equivalence classes).

For $\mathfrak{g}$ of $A(n-1,n-1)$ type, we change (ii) to (ii′):

(ii′) We check, for a certain choice of (S,Π), that $\mathrm{supp}(X)\subset(\rho-Q^{+})$ and that for $\xi:=\sum_{\beta\in S}\beta$, the coefficients of $e^{\rho-s\xi}$ in X and in Re^{ρ} are equal for each $s\in\mathbb{Z}_{\geq 0}$.

The choices of (S,Π) in (ii), (iii) are the same only for $A(m,n)$ case.

4.4 Why (i)–(iii) Imply (1) Let us show that (i)–(iii) imply the denominator identity $X = Re^{\rho}$. Indeed, assume that $X - Re^{\rho} \neq 0$.

Since $WS \subset \Delta_1$, $X - Re^{\rho}$ is a rational function with the denominator of the form $\prod_{\beta\in\Delta_1^+}(1+e^{-\beta})$. We write

$$X - Re^{\rho} = \frac{Y}{\prod_{\beta\in\Delta_1^+}(1+e^{-\beta})} = \frac{Ye^{\rho_1}}{\prod_{\beta\in\Delta_1^+}(e^{\beta/2}+e^{-\beta/2})},$$

where $Y \in \mathbb{Q}[e^{\nu}, \nu \in V]$. We have

$$Re^{\rho} = \frac{\prod_{\alpha\in\Delta_0^+}(e^{\alpha/2}-e^{-\alpha/2})}{\prod_{\alpha\in\Delta_1^+}(e^{\alpha/2}+e^{-\alpha/2})},$$

and the latter expression is W-skew-invariant, since its enumerator is W-skew-invariant and its denominator is W-invariant. Combining (i) and (iii), we obtain that $X - Re^{\rho}$ is W-skew-invariant. Thus, Ye^{ρ_1} is a W-skew-invariant element of $\mathbb{Q}[e^{\nu}, \nu \in V]$, and so $\operatorname{supp}(Ye^{\rho_1})$ is a union of regular orbits.

Observe that $\operatorname{supp}(Re^{\rho}) \subset (\rho - Q^+)$ and that the coefficient of e^{ρ} in Re^{ρ} is 1. Using (i), (ii), we get $\operatorname{supp}(X - Re^{\rho}) \subset (\rho - Q^+) \setminus \{\rho\}$. Note that the sets of maximal elements in $\operatorname{supp}(Y)$ and in $\operatorname{supp}(X - Re^{\rho})$ coincide. Thus, $\operatorname{supp} Y \subset (\rho - Q^+) \setminus \{\rho\}$, that is $\operatorname{supp}(Ye^{\rho_1}) \subset (\rho_0 - Q^+) \setminus \{\rho_0\}$. Hence, $\operatorname{supp}(Ye^{\rho_1})$ is a union of regular orbits lying in $(\rho_0 - Q^+) \setminus \{\rho_0\}$.

By 5.2, for $\mathfrak{g} \neq \mathfrak{gl}(n|n)$, the set $(\rho_0 - Q^+) \setminus \{\rho_0\}$ does not contain regular W-orbit, a contradiction.

Let $\mathfrak{g} = \mathfrak{gl}(n|n)$. Choose Π as in 7.3. By 5.2 the regular orbits in $(\rho_0 - Q^+)$ are of the form $W(\rho_0 - s\xi)$ with $s \in \mathbb{Z}_{\geq 0}$ and $\xi = \sum_{\beta\in S}\beta$. We have $W\xi = \xi$ and $w\rho_0 \leq \rho_0$, so $\rho_0 - s\xi$ is the maximal element in its W-orbit. Thus a maximal element in $\operatorname{supp} Ye^{\rho_1}$ is of the form $\rho_0 - s\xi$, so a maximal element in $\operatorname{supp} Y$ is $\rho - s\xi$. Then, by the above, $\rho - s\xi \in \operatorname{supp}(X - Re^{\rho})$, which contradicts (ii$'$).

5 Regular Orbits

5.1 Let $\mathfrak{g}$ be a reductive finite-dimensional Lie algebra, let W be its Weyl group, let Π be its set of simple roots, and let $\Pi^{\vee}$ be the set of simple coroots. For ρ, defined as above, we have $\langle\rho, \alpha^{\vee}\rangle = 1$ for each $\alpha \in \Pi$. Set

$$Q_{\mathbb{Q}} = \sum_{\alpha\in\Pi}\mathbb{Q}\alpha, \qquad Q_{\mathbb{Q}}^+ := \sum_{\alpha\in\Pi}\mathbb{Q}_{\geq 0}\alpha.$$

As above, we define a partial order on $\mathfrak{h}_{\mathbb{R}}^*$ by the formula $\mu \leq \nu$ if $(\nu - \mu) \in \sum_{\alpha\in\Delta_+}\mathbb{R}_{\geq 0}\alpha$.

Let $P \subset \mathfrak{h}_{\mathbb{R}}^*$ be the weight lattice of $\mathfrak{g}$, i.e. $\nu \in P$ iff $\langle\nu, \alpha^{\vee}\rangle \in \mathbb{Z}$ for any $\alpha \in \Pi$. Let P^+ be the positive part of P, i.e. $\nu \in P^+$ iff $\langle\nu, \alpha^{\vee}\rangle \in \mathbb{Z}_{\geq 0}$ for any $\alpha \in \Pi$. We have $P \subset Q_{\mathbb{Q}}$.

5.1.1. Lemma

(i) $P = WP^+$.
(ii) *For any* $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$, *the stabilizer of* λ *in* W *is either trivial or contains a reflection.*
(iii) *A regular orbit in* P *intersects with the set* $\rho + Q^+_{\mathbb{Q}}$.

Proof The group W is finite. Take $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ and let $\lambda' = w\lambda$ be a maximal element in the orbit $W\lambda$. Since λ' is maximal, $\langle \lambda', \alpha^\vee \rangle \geq 0$ for each $\alpha \in \Pi$.

For $\lambda \in P$, we have $\lambda' \in P$, so $\langle \lambda', \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for each $\alpha \in \Pi$, that is $\lambda' \in P^+$, and hence (i).

For (ii), note that, if $\langle \lambda', \alpha^\vee \rangle = 0$ for some $\alpha \in \Pi$, then $s_\alpha \in \mathrm{Stab}_W \lambda'$, so $s_{w^{-1}\alpha} \in \mathrm{Stab}_W \lambda$. Assume that $\langle \lambda', \alpha^\vee \rangle > 0$ for all $\alpha \in \Pi$. Take $y \in W$, $y \neq \mathrm{id}$ and write $y = y' s_\alpha$ for $\alpha \in \Pi$, where the length of $y' \in W$ is less than the length of y. Then $y'\alpha \in \Delta_+$ (see, for instance, [J], A.1.1). We have $y'\lambda' - y's_\alpha\lambda' = \langle \lambda', \alpha^\vee \rangle (y'\alpha) > 0$, so $y'\lambda' > y's_\alpha\lambda'$. Now (ii) follows by the induction on the length of y.

For (iii), assume that λ' is regular, that is $\langle \lambda', \alpha^\vee \rangle > 0$ for all $\alpha \in \Pi$. Since $\lambda' \in P$, we have $\langle \lambda', \alpha^\vee \rangle \in \mathbb{Z}_{\geq 1}$, so $\langle \lambda - \rho, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Pi$. Write $\lambda - \rho = \sum_{\beta \in \Pi} x_\beta \beta$. For a vector $(y_\beta)_{\beta \in \Pi}$, write $y \geq 0$ if $y_\beta \geq 0$ for each β. The condition $\langle \lambda - \rho, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Pi$ means that $Ax \geq 0$, where $x = (x_\beta)_{\beta \in \Pi}$, and $A = (\langle \alpha^\vee, \beta \rangle)_{\alpha, \beta \in \Pi}$ is the Cartan matrix of $\mathfrak{g}$. From [K3], 4.3, it follows that $Ax \geq 0$ forces $x \geq 0$. Hence, $\lambda - \rho \in Q^+_{\mathbb{Q}}$, as required. ☐

5.2 Now let $\mathfrak{g}$ be a basic simple Lie superalgebra, Q be its root lattice, and Q^+ be the positive part of Q.

5.2.1. Corollary *Let* $\mathfrak{g}$ *be a basic simple Lie superalgebra and* $\mathfrak{g} \neq C(n), A(m,n)$. *A regular orbit in the root lattice* Q *intersects with the set* $\rho_0 + \mathbb{Q}_{\geq 0}\Delta_{+,0}$.

Proof For $\mathfrak{g} \neq C(n), A(m,n)$, we have $\mathbb{Q}\Delta_0 = \mathbb{Q}\Delta$, and the $\mathfrak{g}$-root lattice Q is a subset of weight lattice of $\mathfrak{g}_0$. Thus the assertion follows from Lemma 5.1.1. ☐

5.2.2 Case $\mathfrak{g} = \mathfrak{gl}(m|n)$ For $\mathfrak{gl}(m|n)$, we have

$$\mathbb{Q}\Delta = \mathbb{Q}\Delta_0 \oplus \mathbb{Q}\xi, \quad \text{where } \xi := \sum \varepsilon_i - \frac{m}{n} \sum \delta_j.$$

Choose a set of simple roots as in 7.3.

Lemma *Let* $\mathfrak{g} = \mathfrak{gl}(m|n)$.

(i) *For* $m \neq n$, $W\rho_0$ *is the only regular orbit lying entirely in* $(\rho_0 - Q^+)$.
(ii) *For* $m = n$, *the regular orbits lying entirely in* $(\rho_0 - Q^+)$ *are of the form* $W(\rho_0 - s\xi)$, $s \geq 0$, *where* $\xi = \sum \varepsilon_i - \sum \delta_j$.

Proof Let $\iota : \mathbb{Q}\Delta \to \mathbb{Q}\Delta_0$ be the projection along ξ (i.e. $\mathrm{Ker}\,\iota = \mathbb{Q}\xi$). Since ξ is W-invariant, $w\iota(\lambda) = \iota(w\lambda)$. Let $W\lambda \subset (\rho_0 - Q^+)$ be a regular orbit. Since $\iota(Q)$ lies in the weight lattice of $\mathfrak{g}_0$, $\iota(W\lambda)$ intersects with $\rho_0 + \mathbb{Q}_{\geq 0}\Delta_{+,0}$ by Lemma 5.1.1.

Thus, $W\lambda$ contains a point of the form $\rho_0+\nu+q\xi$, where $\nu\in\mathbb{Q}_{\geq 0}\Delta_{+,0}$ and $q\in\mathbb{Q}$. By the above, $\nu+q\xi\in -Q^+$, so $q\xi\in-\mathbb{Q}_{\geq 0}\Delta_+$.

Consider the case $m>n$. In this case, for any $\mu\in\mathbb{Q}_{\geq 0}\Delta_+$, we have $(\mu,\varepsilon_1)\cdot(\mu,\varepsilon_m)\leq 0$. Since $(\xi,\varepsilon_1)=(\xi,\varepsilon_m)\neq 0$, the inclusion $q\xi\in-\mathbb{Q}_{\geq 0}\Delta_+$ implies $q=0$. Then $\nu\in\mathbb{Q}_{\geq 0}\Delta_{+,0}$ and $\nu=\nu+q\xi\in -Q^+$, so $\nu=0$. Therefore, $W\lambda=W\rho_0$. Hence, $W\rho_0$ is the only regular orbit lying entirely in ρ_0-Q^+. Since $\mathfrak{gl}(m|n)\cong\mathfrak{gl}(n|m)$, this establishes (i).

Consider the case $m=n$. Set $\beta_i:=\varepsilon_i-\delta_i$, $\beta_i':=\delta_i-\varepsilon_{i+1}$. We have $\xi=\sum_{i=1}^n\beta_i$ and $\Pi=\{\beta_i\}_{i=1}^n\cup\{\beta_i'\}_{i=1}^{n-1}$. The simple roots of $\Delta_{+,0}$ are $\varepsilon_i-\varepsilon_{i+1}=\beta_i+\beta_i'$ and $\delta_i-\delta_{i+1}=\beta_i'+\beta_{i+1}$. Thus, $\nu\in\mathbb{Q}_{\geq 0}\Delta_0^+$ takes the form $\nu=\sum_{i=1}^{n-1}b_i(\beta_i+\beta_i')+c_i(\beta_i'+\beta_{i+1})$ with $b_i,c_i\in\mathbb{Q}_{\geq 0}$. By the above, $\nu':=-(\nu+q\xi)\in Q^+$. Therefore, $\nu+\nu'\in\mathbb{Q}\sum_{i=1}^n\beta_i$ and $\nu'\in Q^+$. We readily see that this implies $b_i=c_i=0$, that is $\nu=0$. Since $Q^+\cap\mathbb{Q}\xi=\mathbb{Z}_{\geq 0}\xi$, a regular orbit in ρ_0-Q^+ intersects with the set $\rho_0-\mathbb{Z}_{\geq 0}\xi$, as required. □

5.2.3 Case $C(n)$ Take $\Pi=\{\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_3,\dots,\varepsilon_n-\delta_1,\varepsilon_n+\delta_1\}$. We have $\mathbb{Q}\Delta=\mathbb{Q}\Delta_0\oplus\mathbb{Q}\delta_1$. We claim that $W\rho_0$ is the only regular orbit lying entirely in (ρ_0-Q^+).

Indeed, take a regular orbit lying entirely in (ρ_0-Q^+). Combining the fact that $W\delta_1=\delta_1$ and Corollary 5.1.1, we see that this orbit contains a point of the form $\rho_0+\nu+q\delta_1$, where $\nu\in\mathbb{Q}_{\geq 0}\Delta_0^+$ and $q\in\mathbb{Q}$. Since $-(\nu+q\delta_1)\in Q^+$, we have $q\delta_1\in-\mathbb{Q}_{\geq 0}Q^+$. However, $2\delta_1$ is the difference of two simple roots ($2\delta_1=(\varepsilon_n+\delta_1)-(\varepsilon_n-\delta_1)$) so $\mathbb{Q}\delta_1\cap(-\mathbb{Q}_{\geq 0}Q^+)=\{0\}$, that is $q=0$. The conditions $\nu\in\mathbb{Q}_{\geq 0}\Delta_0^+$, $-(\nu+q\delta_1)\in Q^+$ give $\nu=0$. The claim follows.

6 Step (I)

Consider the following operations with the admissible pairs (S,Π). First-type operations are the odd reflections $(S,\Pi)\mapsto(s_\beta S,s_\beta\Pi)=(S\setminus\{\beta\}\cup\{-\beta\},s_\beta\Pi)$ with respect to an element of $\beta\in S$ (see 6.2). By 6.2, these odd reflections preserve the expressions X, Re^ρ. Second-type operations are the operations $(S,\Pi)\mapsto(S',\Pi)$ described in Lemma 6.3, where it is shown that these operations also preserve the expressions X, Re^ρ. Consider the equivalence relation on the set of admissible pairs (S,Π) generated by these operations. In 6.4 we will show that there are two equivalence classes for $D(m,n)$, $m>n$, and one equivalence class for other cases. In 6.5 we will show that if (S,Π), (S,Π') are admissible pairs, then $\Pi=\Pi'$ (on the other hand, there are admissible pairs (S,Π), (S',Π) with $S\neq S'$, see Lemma 6.3).

6.1 Notation Let us introduce the operator $F:\mathcal{R}\to\mathcal{R}$ by

$$F(Y):=\sum_{w\in W^\#}\operatorname{sgn}(w)wY.$$

Clearly, $F(wY) = \mathrm{sgn}(w)F(Y)$ for $w \in W^{\#}$, so $F(Y) = 0$ if $wY = Y$ for some $w \in W^{\#}$ with $\mathrm{sgn}(w) = -1$.

For an admissible pair (S, Π), introduce

$$Y(S, \Pi) := \frac{e^{\rho(\Pi)}}{\prod_{\beta \in S}(1 + e^{-\beta})}, \qquad X(S, \Pi) := F\big(Y(S, \Pi)\big),$$

where $\rho(\Pi)$ is the element ρ defined for given Π. Note that $X = X(S, \Pi)$ for the corresponding pair (S, Π).

6.2 Odd Reflections Recall the notion of odd reflections, see [S]. Let Π be a set of simple roots, and $\Delta_+(\Pi)$ be the corresponding set of positive roots. Fix a simple isotropic root $\beta \in \Pi$ and set

$$s_\beta(\Delta_+) := \Delta_+(\Pi) \setminus \{\beta\} \cup \{-\beta\}.$$

For each $P \subset \Pi$, set $s_\beta(P) := \{s_\beta(\alpha) |\ \alpha \in P\}$, where

$$\text{for } \alpha \in \Pi, \quad s_\beta(\alpha) := \begin{cases} -\alpha & \text{if } \alpha = \beta, \\ \alpha & \text{if } (\alpha, \beta) = 0,\ \alpha \neq \beta, \\ \alpha + \beta & \text{if } (\alpha, \beta) \neq 0. \end{cases}$$

By [S], $s_\beta(\Delta_+)$ is a set of positive roots, and the corresponding set of simple roots is $s_\beta(\Pi)$. Clearly, $\rho(s_\beta(\Pi)) = \rho(\Pi) + \beta$.

Let (S, Π) be an admissible pair. Take $\beta \in S$. Then for any $\beta' \in S \setminus \{\beta\}$, we have $s_\beta(\beta') = \beta'$, so $s_\beta(S) = (S \setminus \{\beta\}) \cup \{-\beta\}$. Clearly, the pair $(s_\beta(S), s_\beta(\Pi))$ is admissible. Since $\rho(s_\beta(\Pi)) = \rho(\Pi) + \beta$, we have $Y(S, \Pi) = Y(s_\beta(S), s_\beta(\Pi))$.

6.3. Lemma *Assume that γ, γ' are isotropic roots such that*

$$\gamma \in S, \quad \gamma' \in \Pi, \quad \gamma + \gamma' \in \Delta^{\#}, \quad (\gamma', \beta) = 0 \quad \textit{for each } \beta \in S \setminus \{\gamma\}.$$

Then the pair (S', Π), where $S' := (S \cup \{\gamma'\}) \setminus \{\gamma\}$, is admissible, and $X(S, \Pi) = X(S', \Pi)$.

Proof It is clear that the pair (S', Π) is admissible. Set $\alpha := \gamma + \gamma'$ and let $s_\alpha \in W^{\#}$ be the reflection with respect to the root α. Our assumptions imply that

$$s_\alpha \rho = \rho; \qquad s_\alpha \gamma' = -\gamma; \qquad s_\alpha \beta = \beta \quad \text{for } \beta \in S \setminus \{\gamma\}. \tag{4}$$

Therefore, $s_\alpha(Y(S', \Pi)) = Y(S, \Pi)e^{-\gamma}$, that is $F(Y(S', \Pi)) = F(s_\alpha(Y(S, \Pi)e^{-\gamma}))$. Since $F \circ s_\alpha = -F$, the required formula $X(S, \Pi) = X(S', \Pi)$ is equivalent to the equality $F(Y(S, \Pi)(1 + e^{-\gamma})) = 0$, which follows from the fact that the expression

$$Y(S, \Pi)(1 + e^{-\gamma}) = \frac{e^{\rho}}{\prod_{\beta \in S \setminus \{\gamma\}}(1 + e^{-\beta})}$$

is, by (4), s_α-invariant. □

6.4 Equivalence Classes We consider the types $A(m-1,n-1)$, $B(m,n)$, $D(m,n)$ with all possible m,n. We express the roots in terms of linear functions $\xi_1,\ldots,\xi_{m+n}$, see [K1], such that

Δ	$A(m-1,n-1)$	$B(m,n)$	$D(m,n)$
$\Delta_{+,0}$	U	$U'\cup\{\xi_i\}_{i=1}^{m}\cup\{2\xi_i\}_{i=m+1}^{m+n}$	$U'\cup\{2\xi_i\}_{i=m+1}^{m+n}$
Δ_1	$\{\pm(\xi_i-\xi_j)\}_{1\le i\le m<j\le m+n}$	$\{\pm\xi_i\pm\xi_j;$ $\pm\xi_j)\}_{1\le i\le m<j\le m+n}$	$\{\pm\xi_i\pm\xi_j\}_{1\le i\le m<j\le m+n}$

and

$$U:=\{\xi_i-\xi_j \mid 1<i<j\le m \text{ or } m+1\le i<j\le m+n\},$$

$$U':=\{\xi_i\pm\xi_j \mid 1<i<j\le m \text{ or } m+1\le i<j\le m+n\}.$$

6.4.1 Let $f: V\to\mathbb{R}$ be such that $f(\alpha)\neq 0$ for any $\alpha\in\Delta$. Define the decomposition $\Delta=\Delta_+(f)\sqcup\Delta_-(f)$ by $\Delta_+(f):=\{\alpha\in\Delta \mid \langle f,\alpha\rangle>0\}$ and $\Delta_-(f)=-\Delta_+(f)$. Denote by $\Pi(f)$ the set of simple roots for this decomposition.

For a system of simple roots $\Pi\subset\operatorname{span}\{\xi_i\}_{i=1}^{m+n}$, take an element $f_\Pi\in\operatorname{span}\{\xi_i^*\}_{i=1}^{m+n}$ such that $\langle f_\Pi,\alpha\rangle=1$ for each $\alpha\in\Pi$ (the existence of f_Π follows from linear independence of the elements in Π). Note that f_Π is unique if $\mathfrak{g}$ is not of the type $A(m-1,n-1)$; for $A(m-1,n-1)$, we fix f_Π by the additional condition $\min_{1\le i\le m+n}\langle f_\Pi,\xi_i\rangle=1$. In the above notation we have $\Pi=\Pi(f_\Pi)$.

We will use the following properties of f_Π:

(i) $\langle f_\Pi,\alpha\rangle\in\mathbb{Z}\setminus\{0\}$ for all $\alpha\in\Delta$;
(ii) if $\alpha\in\Delta^+$, then $\langle f_\Pi,\alpha\rangle\ge 1$;
(iii) a root $\alpha\in\Delta$ is simple iff $\langle f_\Pi,\alpha\rangle=1$.

Write $f_\Pi=\sum_{i=1}^{m+n}x_i\xi_i^*$. By (i), the x_is are pairwise different; by (ii) , $x_i\pm x_{i+1}>0$ for $i\neq m$ $(1\le i\le m+n-1)$; and by (iii), a root $\xi_i-\xi_j$ is simple iff $x_i-x_j=1$.

6.4.2 For type $A(m-1,n-1)$, all roots are of the form $\xi_i-\xi_j$. In particular, $\{x_i\}_{i=1}^{m+n}=\{1,\ldots,m+n\}$.

Consider the case $B(m,n)$. In this case, for each i, we have $\xi_i\in\Delta_+$, so $x_i\ge 1$. Therefore, by (iii), a simple root cannot be of the form $\pm(\xi_i+\xi_j)$, so $\Pi=\{\xi_{i_1}-\xi_{i_2},\xi_{i_2}-\xi_{i_3},\ldots,\xi_{i_{m+n-1}}-\xi_{i_{m+n}},\xi_{i_{m+n}}\}$ and $\{x_i\}_{i=1}^{m+n}=\{1,\ldots,m+n\}$.

6.4.3 Consider the case $D(m,n)$. Using (i), (ii) and the fact that $2\xi_{m+n}\in\Delta_{+,0}$, we conclude

$$\forall j,\quad 2x_j\in\mathbb{Z};\qquad \forall i\neq j,\quad x_i\pm x_j\in\mathbb{Z}\setminus\{0\};$$

$$x_i-x_j\ge j-i\quad \text{for } i<j\le m \text{ or } m<i<j;$$

$$x_j>0\quad \text{for } j>m;\qquad x_j+x_i>0\quad \text{for } i<j\le m.$$

Recall that, if $-(\xi_p+\xi_q)\in\Pi\cap\Delta_1$, then $\Pi':=s_{-\xi_p-\xi_q}(\Pi)$ contains $\xi_p+\xi_q$. Let us *assume that* $\xi_p+\xi_q\in\Pi\cap\Delta_1$, $p<q$, that is $x_p+x_q=1$, $p\le m<q$. If $p<m$,

then $x_p+x_q > \pm(x_m+x_q)$ (because $x_p \pm x_m, 2x_q > 0$), so $x_p+x_q > |x_m+x_q| \geq 1$, a contradiction. Hence $p=m$; in particular,

$$\pm(\xi_p+\xi_q) \in \Pi \cap \Delta_1, \quad p<q \Longrightarrow p=m. \tag{5}$$

Since $2x_q \in \mathbb{Z}_{>0}$, $x_p+x_q=1$ and $x_q \neq x_m$, we have $x_m \leq 0$. Therefore the assumption implies

$$x_m+x_q=1 \;\&\; q>m \;\&\; x_m \leq 0.$$

Since $\xi_{m-1}-\xi_m \in \Delta^+$, there exists a simple root of the form $\pm\xi_s-\xi_m$ $(s \neq m)$.

First, consider the case where $\xi_s-\xi_m \in \Pi$, that is $x_s-x_m=1$. Since $x_m \leq 0$, we have $x_s+x_m \leq x_s-x_m=1$, so $x_s+x_m<0$, that is $-(\xi_s+\xi_m) \in \Delta_+$. Therefore, $s>m$ and $x_q+x_s=2$, because $x_q+x_m=x_s-x_m=1$. Since $x_{m+n-i} \geq 1/2+i$ for $i<n$, we conclude that either $x_q=3/2, x_m=-1/2, x_s=1/2$, which contradicts to $x_m+x_s \neq 0$, or $q=s=m+n$, $x_{m+n}=1$, $x_m=0$. Hence, $\xi_s-\xi_m \in \Pi$ implies $q=m+n$, $x_{m+n}=1$, $x_m=0$ and $\xi_{m+n} \pm \xi_m \in \Pi$.

Now consider the case where $-\xi_s-\xi_m \in \Pi$. Then $s>m$ and $x_m+x_q=-x_m-x_s=1$ so $x_q-x_s=2$. Since $x_q-x_{q+i} \geq i$, we have $s=q+1$ or $s=q+2$. If $s=q+2$, then we have $2=x_q-x_{q+2}=(x_q-x_{q+1})+(x_{q+1}-x_{q+2})$, that is $x_q-x_{q+1}=1$, so $x_m+x_{q+1}=0$, a contradiction. Hence $s=q+1$, that is $-x_m-x_{q+1}=1$. For $i<m$, we have $0<x_i+x_m=-x_{q+1}-1+x_i$, so $1<x_i-x_{q+1}$. Therefore, for $i<m$ and $t \geq q+1$, we have $x_i+x_t > x_i-x_t > 1$, so the roots $\pm\xi_i \pm \xi_t$ are not simple. Assume that $m \geq n$ and (S,Π) is an admissible pair. Then, for each $m<t \leq m+n$, there exists $i_t<m$ such that one of the roots $\pm\xi_t \pm \xi_{i_t}$ is simple (and lies in S) and $i_t \neq i_p$ for $t \neq p$. By the above, $i_{q+1}=m$, and there is no suitable i_t for $t>q+1$. Hence, $q+1=m+n$ and $(-\xi_m-\xi_{m+n}) \in S$, that is $\xi_m+\xi_{m+n} \in s_{-\xi_m-\xi_{m+n}}S$.

6.4.4 We conclude that if (S,Π) is an admissible pair for $\mathfrak{g}=D(m,n)$, then one of the following possibilities hold: either all elements of S are of the form $\xi_i-\xi_j$, or all elements of S except one are of the form $\xi_i-\xi_j$, and this exceptional one is β, with one of the following possibilities:

(1) $\beta=\xi_m+\xi_{m+n}$ and $\xi_{m+n}-\xi_m \in \Pi$;
(2) $m<n$ and $\beta=\xi_m+\xi_s$, $m<s<m+n$, $-\xi_{s+1}-\xi_m \in \Pi$;
(3) $\beta:=-(\xi_m+\xi_s)$, and the pair $(s_\beta S, s_\beta \Pi)$ is one of whose described in (1)–(2).

6.4.5 Consider the case $D(m,n)$, $n \geq m$. Then $\Delta^{\#}=\{\xi_i \pm \xi_j; 2\xi_i\}_{i=m+1}^{m+n}$. Let (S,Π) be an admissible pair. If $\xi_m+\xi_s \in S$ for $s<m+n$, then, by the above, $-(\xi_m+\xi_{s+1}) \in \Pi$, and the pair (S,Π) is equivalent to the pair $((S \setminus \{\xi_m+\xi_s\}) \cup \{-\xi_m-\xi_{s+1}\}, \Pi)$, which is equivalent to the pair $((S \setminus \{\xi_m+\xi_s\}) \cup \{\xi_m+\xi_{s+1}\}, s_{-\xi_m-\xi_{s+1}}\Pi)$. Thus a pair (S,Π) with $\xi_m+\xi_s \in S$ is equivalent to a pair (S',Π') with $\xi_m+\xi_{m+n} \in S$. If $\xi_m+\xi_{m+n} \in S$, then, by the above, $\xi_{m+n}-\xi_m \in \Pi$, and the pair (S,Π) is equivalent to the pair $((S \setminus \{\xi_{m+n}+\xi_m\}) \cup \{\xi_{m+n}-\xi_m\}, \Pi)$. We conclude that any pair (S,Π) is equivalent to a pair (S',Π'), where $S'=\{\xi_i-\xi_{i_j}\}_{i=1}^m$.

Consider the case $D(m,n)$, $m>n$. Then $\Delta^{\#}=\{\xi_i\pm\xi_j\}_{i=1}^m$. By the above, any pair (S,Π) is equivalent either to a pair (S',Π'), where $S'=\{\xi_i-\xi_{i_j}\}_{i=1}^n$, or to a pair (S',Π'), where $S'=\{\xi_i-\xi_{i_j}\}_{i=1}^{n-1}\cup\{\xi_{m+n}+\xi_m\}$ and $\xi_{m+n}-\xi_m\in\Pi'$.

6.4.6 Let (S,Π) be an admissible pair. We conclude that any pair (S,Π) is equivalent to a pair (S',Π'), where either $S'=\{\xi_i-\xi_{j_i}\}_{i=1}^{\min(m,n)}$, or, for $D(m,n)$, $m>n$, $S'=\{\xi_i-\xi_{i_j}\}_{i=1}^{n-1}\cup\{\xi_{m+n}+\xi_m\}$ and $\xi_{m+n}-\xi_m\in\Pi'$.

6.5 Fix a set of simple roots of $\mathfrak{g}$ and construct f_Π as in 6.4. We mark the points x_i on the real line by a's and b's in one of the following ways:

(M) mark x_i by a (resp., by b) if $1\leq i\leq m$ (resp., if $m<i\leq m+n$);
(N) mark x_i by b (resp., by a) if $1\leq i\leq m$ (resp., if $m<i\leq m+n$).

We use the marking (M) if $\Delta^{\#}$ lies in the span of $\{\xi_i\}_{i=1}^m$ and the marking (N) if $\Delta^{\#}$ lies in the span of $\{\xi_i\}_{i=m+1}^{m+n}$. Note that in all cases the number of as is not smaller than the number of b's.

We fix $\Delta^{\#}$ and an admissible pair (S,Π) such that $S=\{\xi_i-\xi_{j_i}\}_{i=1}^{\min(m,n)}$ or, for $D(m,n)$, $m>n$, $S=\{\xi_i-\xi_{i_j}\}_{i=1}^{n-1}\cup\{\xi_{m+n}+\xi_m\}$.

If $\xi_i-\xi_j\in S$ (resp., $\xi_i+\xi_j\in S$) we draw a bow $\smile$ (resp., $\frown$) between the points x_i and x_j. Observe that the points connected by a bow are neighbors and they are marked by different letters (a and b). We say that a marked point is a vertex if it is a vertex of a bow. Note that the bows do not have common vertices and that all points marked by b are vertices.

From now one we consider the diagrams which are sequences of a's and b's endowed with the bows (we do not care about the values of x_i). For example, for $\mathfrak{g}=A(4,1)$ and $\Pi=\{\xi_1-\xi_2;\xi_2-\xi_6;\xi_6-\xi_7;\xi_7-\xi_3;\xi_3-\xi_4;\xi_4-\xi_5\}$, we choose $f=\xi_5^*+2\xi_4^*+3\xi_3^*+4\xi_7^*+5\xi_6^*+6\xi_2^*+7\xi_1^*$, and taking $S=\{\xi_2-\xi_6;\xi_7-\xi_3\}$, we obtain the diagram $aaa\smile bb\smile aa$; for $\mathfrak{g}=B(2,2)$ and $\Pi=\{\xi_1-\xi_3;\xi_3-\xi_2;\xi_2-\xi_4;\xi_4\}$, we choose $f=\xi_4^*+2\xi_2^*+3\xi_3^*+4\xi_1^*$, and taking $S=\{\xi_1-\xi_3;\xi_2-\xi_4\}$, we obtain the diagram $a\smile ba\smile b$ for the marking (N) and $b\smile ab\smile a$ for the marking (M). Observe that a diagram containing $\frown$ appears only in the case $D(m,n)$, $m>n$, and such a diagram starts from $a\frown b$, because $\xi_{m+n}+\xi_m\in S$ forces $\xi_{m+n}-\xi_m\in\Pi$, which implies $x_m=0$, $x_{m+n}=1$, $x_i>0$ for all $i\neq m$.

6.5.1 Let us see how the odd reflections and the operations of the second type, introduced in Lemma 6.3, change our diagrams.

Recall that $\xi_i-\xi_j\in\Delta_+$ iff $x_i>x_j$. For an odd simple root β, we have $s_\beta(\Delta_+)=(\Delta_+\setminus\{\beta\})\cup\{-\beta\}$, so the order of x_i's for $s_{\xi_p-\xi_q}(\Delta_+)$ is obtained from the order of x_i's for Δ_+ by the interchange of x_p and x_q (if $\xi_p-\xi_q\in\Pi\cap\Delta_1$). Therefore the odd reflection with respect to $\xi_p-\xi_q\in S$ corresponds to the following operation with the diagram where we interchange the vertices (i.e. the marks a,b) of the corresponding bow:

$$\ldots a\smile b\ldots\mapsto\ldots b\smile a\ldots;\qquad \ldots b\smile a\ldots\mapsto\ldots a\smile b\ldots$$

If the diagram has a part $a \smile ba$ (resp. $ab \smile a$), where the last (resp. the first) sign a is not a vertex, and x_i, x_j, x_k are the corresponding points, then the quadrapole (S, Π), $\gamma := \xi_i - \xi_j$, $\gamma' := \xi_j - \xi_k$ (resp. $\gamma' := \xi_i - \xi_j$, $\gamma := \xi_j - \xi_k$) satisfies the assumptions of Lemma 6.3. The operation $(S, \Pi) \mapsto (S', \Pi)$, where $S' := (S \setminus \{\gamma\}) \cup \{\gamma'\}$, corresponds to the following operation with our diagram: $a \smile ba \mapsto ab \smile a$ (resp. $ab \smile a \mapsto a \smile ba$). Hence we can perform the operation of the second type if a, b, a are neighboring points, b is connected by $\smile$ with one of a's, and another a is not a vertex; in this case, we remove the bow and connect b with another a:

$$\ldots ab \smile a \ldots \mapsto \ldots a \smile ba \ldots; \qquad \ldots a \smile ba \ldots \mapsto \ldots ab \smile a \ldots$$

Since both our operations are involutions, we can consider the orbit of a given diagram with respect to the action of the group generated by these operations. Let us show that all diagrams without $\frown$ lie in the same orbit. Indeed, using the operations $a \smile b \mapsto b \smile a$ and $ab \smile a \mapsto a \smile ba$, we put b to the first place, so our new diagram starts from $b \smile a$. Then we do the same with the rest of the diagram, and so on. Finally, we obtain the diagram of the form $b \smile ab \smile a \ldots b \smile a \ldots a$. By the same argument, all the diagrams starting from $b \frown a$ lie in the same orbit. Hence, for $\mathfrak{g} \neq D(m,n)$, $m > n$, all diagrams lie in the same orbit, and for $\mathfrak{g} = D(m,n)$, $m > n$, there are two orbits: the diagrams with $\frown$ and the diagrams without $\frown$.

6.5.2 We claim that if $(S, \Pi), (S, \Pi')$ are admissible pairs, then $\Pi = \Pi'$. It is clear that if the claim is valid for some S, then it is valid for all S' such that (S, Π) is equivalent to (S', Π'). Thus it is enough to verify the claim for any representative of the orbit. Each S determines the diagram, and the diagram determines the order of x_i's (for instance, the diagram $b \smile ab \smile a \ldots b \smile aa \ldots a$ gives $x_{m+n} < x_m < x_{m+n-1} < x_{m-1} < \cdots < x_{m+1} < x_{m-n+1} < x_{m-n} < \cdots < x_1$ for the marking (M) and $x_m < x_{m+n} < x_{m-1} < x_{m+n-1} < \cdots < x_1 < x_{n+1} < x_n < \cdots < x_{m+1}$ for the marking (N)). For $A(m-1, n-1)$, $B(m,n)$, we have $\{x_i\}_{i=1}^{m+n} = \{1, \ldots, m+n\}$, by 6.4.2, so each diagram determines Π.

For $D(m,n)$, $m > n$, the above diagram means that $-\xi_{m+n} + \xi_m, -\xi_m + \xi_{m+n-1}, \ldots \in \Pi$. Since $2\xi_{m+n} \in \Delta_+$, we conclude that $2\xi_{m+n} \in \Pi$, that is $x_{m+n} = \frac{1}{2}$ and $\{x_i\}_{i=1}^{m+n} = \{\frac{1}{2}, \ldots, m+n-\frac{1}{2}\}$. Hence the above diagram determines Π.

For $D(m,n)$, $m \leq n$, the same reasoning shows that the diagram $a \smile bb \smile a \ldots b \smile aa \ldots a$ determines Π.

It remains to consider the case $D(m,n)$, $m > n$, and the diagram $a \frown bb \smile ab \smile a \ldots b \smile aa \ldots a$. In this case, $\xi_m + \xi_{m+n} \in S$, so, by the above, $x_m = 0$ and $\{x_i\}_{i=1}^{m+n} = \{0, \ldots, m+n-1\}$. Thus the diagram determines Π.

6.5.3 Consider the case $\mathfrak{g} \neq D(m,n)$, $m > n$. Fix an admissible pair (S, Π). By 6.5, any admissible pair (S', Π') is equivalent to an admissible pair (S, Π''). By 6.5.2 we have $\Pi = \Pi''$, so (S', Π') is equivalent to (S, Π).

Consider the case $\mathfrak{g} = D(m,n)$, $m > n$. Fix admissible pairs (S, Π), (S', Π'), where S consists of the roots of the form $\xi_i - \xi_j$, and S' contains $\xi_m + \xi_{m+n}$. Arguing as above, we conclude that any admissible pair (S'', Π'') is equivalent either to (S, Π) or to (S', Π').

6.6. Corollary

(i) *If* (S, Π), (S, Π') *are admissible pairs, then* $\Pi = \Pi'$.
(ii) *For* $\mathfrak{g} \neq D(m,n)$, $m > n$, *there is one equivalence class of the pairs* (S, Π), *and the left-hand (resp., right-hand) side of* (1) *is the same for all admissible pairs* (S, Π).
(iii) *For* $\mathfrak{g} = D(m,n)$, $m > n$, *there are two equivalence classes of the pairs* (S, Π). *In the first class,* S *consists of the elements of the form* $\pm(\xi_i - \xi_j)$, *and in the second class,* S *contains a unique element of the form* $\pm(\xi_i + \xi_j)$. *The left-hand (resp., right-hand) side of* (1) *is the same for all admissible pairs* (S, Π) *belonging to the same class.*

7 Steps (ii), (ii′)

7.1 Assume that

$$\begin{array}{l} \forall \alpha \in \Pi, \quad (\alpha,\alpha) \geq 0; \\ W^{\#} \text{ is generated by the set of reflections } \{s_\alpha \mid (\alpha,\alpha) > 0\}. \end{array} \tag{6}$$

We start with the following lemma.

7.1.1. Lemma *We have*:

(i) $\rho \geq w\rho$ *for all* $w \in W^{\#}$;
(ii) *the stabilizer of* ρ *in* $W^{\#}$ *is generated by the set* $\{s_\alpha \mid (\alpha,\alpha) > 0,\ (\alpha,\rho) = 0\}$.

Proof Since $(\alpha,\rho) = \frac{1}{2}(\alpha,\alpha) \geq 0$ for all $\alpha \in \Pi$, we have $(\beta,\rho) \geq 0$ for all $\beta \in \Delta_+$. Take $w \in W^{\#}$, $w \neq \text{id}$. We have $w = w's_\beta$, where $w' \in W^{\#}$ and $\beta \in \Delta_+$ are such that the length of w is greater than the length of w' and $w'\beta \in \Delta_+$ (see, for example, [J, A.1]). We have

$$\rho - w\rho = \rho - w'\rho + \frac{2(\rho,\beta)}{(\beta,\beta)} \cdot (w'\beta).$$

Now (i) follows by induction on the length of w, since $(\rho,\beta) \geq 0, (\beta,\beta) > 0$, $w'\beta \in \Delta_+$. For (ii), note that $\rho \geq w'\rho$ by (i), thus $w\rho = \rho$ forces $\rho = w'\rho$ and $(\rho,\beta) = 0$. Hence (ii) also follows by induction on the length of w. □

7.1.2 Retain the notation of (3). For $w \in W^{\#}$, we have $-\varphi(w), |w|\mu \in Q^+$, by definition, and $w\rho \leq \rho$, by Lemma 7.1.1. Therefore, $\varphi(w) - |w|\mu + w\rho \leq \rho$, and the equality means that $\mu = 0$, $w\rho = \rho$ and $\varphi(w) = 0$, that is $wS \subset \Delta_+$. The above inequality gives $\text{supp}(X) \subset (\rho - Q^+)$. Moreover, by the above, the coefficient of e^ρ in the expansion of X is equal to

$$\sum_{w \in W^{\#}: w\rho = \rho,\ wS \subset \Delta_+} \text{sgn}(w).$$

7.2 Root Systems Recall that we consider all choices of Δ_+ with a fixed $\Delta_{+,0}$. From now on we *assume that* $m \geq n$ (we consider the types $A(m,n)$, $B(m,n)$, $B(n,m)$, $D(m,n)$, $D(n,m)$), and we embed our root systems in the standard lattices spanned by $\{\varepsilon_i, \delta_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ chosen in such a way that $\Delta^{\#} = \Delta_0 \cap \operatorname{span}\{\varepsilon_i\}_{i=1}^m$. More precisely, for $A(m,n)$, $m \geq n$, we take

$$\Delta_{+,0} = \{\varepsilon_i - \varepsilon_{i'}; \delta_j - \delta_{j'} | 1 \leq i < i' \leq m, 1 \leq j < j' \leq n\}, \qquad \Delta_1 = \left\{\pm(\varepsilon_i - \delta_j)\right\};$$

for other cases, we put

$$U' := \{\varepsilon_i \pm \varepsilon_{i'}; \delta_j \pm \delta_{j'} | 1 \leq i < i' \leq m, 1 \leq j < j' \leq n\}$$

and then

$$\Delta_{+,0} = U' \cup \{\varepsilon_i\}_{i=1}^m \cup \{2\delta_j\}_{j=1}^n, \qquad \Delta_1 = \{\pm\varepsilon_i \pm \delta_j, \pm\delta_j\}$$

for $B(m,n), m > n$ and $B(n,n)$, $\Delta^{\#} = B(n)$;

$$\Delta_{+,0} = U' \cup \{2\varepsilon_i\}_{i=1}^m \cup \{\delta_j\}_{j=1}^n, \qquad \Delta_1 = \{\pm\varepsilon_i \pm \delta_j, \pm\varepsilon_i\}$$

for $B(n,m),\ m > n$ and $B(n,n),\ \Delta^{\#} = C(n)$;

$$\Delta_{+,0} = U' \cup \{2\delta_j\}_{j=1}^n, \qquad \Delta_1 = \{\pm\varepsilon_i \pm \delta_j\} \quad \text{for } D(m,n),\ m > n;$$

$$\Delta_{+,0} = U' \cup \{2\varepsilon_i\}_{i=1}^m, \qquad \Delta_1 = \{\pm\varepsilon_i \pm \delta_j\} \quad \text{for } D(n,m),\ m \geq n.$$

We normalize the form $(-,-)$ by the condition $(\varepsilon_i, \varepsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$. We have $\Delta^{\#} = \Delta_0 \cap \operatorname{span}\{\varepsilon_i\}_{i=1}^m = \{\alpha \in \Delta_0 |\ (\alpha, \alpha) > 0\}$.

7.3 Choice of (S, Π) For the case $B(n,n)$, set

$$S := \{\delta_i - \varepsilon_i\}, \qquad \Pi := \{\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \delta_2 - \varepsilon_2, \ldots, \delta_n - \varepsilon_n, \varepsilon_n\}$$

(the root ε_n may be even or odd, depending on the choice of $\Delta^{\#}$). For other cases, set

$$S := \{\varepsilon_i - \delta_i\}_{i=1}^n.$$

In order to describe Π, introduce

$$P := \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \varepsilon_2 - \delta_2, \delta_2 - \varepsilon_3, \ldots, \varepsilon_n - \delta_n,$$
$$\delta_n - \varepsilon_{n+1}, \varepsilon_{n+1} - \varepsilon_{n+2}, \ldots, \varepsilon_{m-1} - \varepsilon_m\}$$

and set

Δ	$A(m-1,n-1)$	$B(m,n)$, $B(n,m)$, $n > m$	$D(n,m)$, $m > n$	$D(m,n)$, $m > n$
Π	P	$P \cup \{\varepsilon_m\}$	$P \cup \{2\varepsilon_m\}$	$P \cup \{\varepsilon_{m-1} + \varepsilon_m\}$

$$\Pi := \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \varepsilon_2 - \delta_2, \delta_2 - \varepsilon_3, \ldots, \varepsilon_n - \delta_n, \varepsilon_n + \delta_n\} \quad \text{for } D(n,n).$$

Note that assumptions (6) hold.

7.4 Step (ii): the Case $(\varepsilon_m+\delta_n)\notin\Pi$ By 7.1.2, $\operatorname{supp}(X)\subset(\rho-Q^+)$, and the coefficient of e^ρ in the expansion of X is $\sum_{w\in W^\#:w\rho=\rho,wS\subset\Delta_+}\operatorname{sgn}(w)$. Thus it is enough to show that

$$w\in W^\# \quad \text{s.t. } wS\subset\Delta_+,\ w\rho=\rho\Longrightarrow w=\mathrm{id}. \tag{7}$$

7.4.1 Since $(\alpha,\alpha)\geq 0$ for each $\alpha\in\Pi$, we have $(\rho,\beta)=0$ for $\beta\in\Delta$ iff β is a linear combination of isotropic simple roots.

Consider the case $\mathfrak{g}\neq D(n,n)$ (one has $\rho=0$ for $D(n,n)$). In this case, $(\rho,\beta)=0$ for $\beta\in\Delta_+^\#$ forces $\beta=\varepsilon_i-\varepsilon_j$ for $i<j\leq\min(m,n+1)$. From Lemma 7.1.1 we conclude

$$\text{for } \mathfrak{g}\neq D(n,n),\qquad \operatorname{Stab}_{W^\#}\rho=\begin{cases} S_n & \text{if } m=n,\\ S_{n+1} & \text{if } m>n,\end{cases} \tag{8}$$

where $S_k\subset W^\#$ is the symmetric group consisting of the permutations of $\varepsilon_1,\dots,\varepsilon_k$ (that is for $w\in S_k$, we have $w\varepsilon_i=\varepsilon_{j_i}$ and $j_i=i$ for $i>k$).

7.4.2 Take $w\in\operatorname{Stab}_{W^\#}\rho$ such that $wS\subset\Delta_+$. Let us show that $w=\mathrm{id}$.

Consider the case where $\mathfrak{g}\neq B(n,n), D(n,n)$. Then $S=\{\varepsilon_i-\delta_i\}_{i=1}^n$. Combining (8) and the fact that $\varepsilon_j-\delta_i\in\Delta_+$ iff $j\leq i$, we conclude that $w\varepsilon_i=\varepsilon_{j_i}$ for $j_i\leq i$ if $i\leq n$ and $j_i=i$ for $i>n+1$. Hence $w=\mathrm{id}$, as required.

For $\mathfrak{g}=D(n,n)$, we have $S=\{\varepsilon_i-\delta_i\}_{i=1}^n$. Since $-\varepsilon_i-\delta_j\notin\Delta_+$ for all i,j, the condition $wS\subset\Delta_+$ forces $w\in S_n$ (see 7.4.1 for notation). Repeating the above argument, we obtain $w=\mathrm{id}$.

For $\mathfrak{g}=B(n,n)$, we have $S=\{\delta_i-\varepsilon_i\}_{i=1}^n$. Combining (8) and the fact that $\delta_i-\varepsilon_j\in\Delta_+$ iff $j\geq i$, we obtain for all $i=1,\dots,n$ that $w\varepsilon_i=\varepsilon_{j_i}$ for some $j_i\leq i$. Hence $w=\mathrm{id}$, as required. This establishes (7) and (ii) for the case $(\varepsilon_m+\delta_n)\notin\Pi$.

7.5 Step (ii): the Case $(\varepsilon_m+\delta_n)\in\Pi$ Consider the case $D(m,n)$, $m>n$, $(\varepsilon_m+\delta_n)\in\Pi$. We retain notation of 7.2 and choose a new pair (S,Π): $S=\{\varepsilon_i-\delta_i\}_{i=1}^{n-1}\cup\{\varepsilon_m+\delta_n\}$ and

$$\begin{aligned}\Pi :=&\{\varepsilon_1-\delta_1,\delta_1-\varepsilon_2,\dots,\delta_{n-1}-\varepsilon_n\}\cup\{\varepsilon_i-\varepsilon_{i+1}\}_{i=n}^{m-2}\\ &\cup\{\varepsilon_{m-1}-\delta_n,\delta_n-\varepsilon_m,\delta_n+\varepsilon_m\}.\end{aligned}$$

Assumptions (6) are satisfied. By 7.1.2, the coefficient of e^ρ in the expansion of X is equal to $\sum_{w\in A}\operatorname{sgn}(w)$, where $A:=\{w\in\operatorname{Stab}_{W^\#}\rho|\ wS\subset\Delta_+\}$. Let us show that $A=\{\mathrm{id}, s_{\varepsilon_{m-1}-\varepsilon_m}, s_{\varepsilon_{m-1}-\varepsilon_m}s_{\varepsilon_{m-1}+\varepsilon_m}\}$; this implies that the coefficient of e^ρ in the expansion of X is equal to 1.

Take $w\in W^\#$ such that $wS\subset\Delta_+$. Note that $-\varepsilon_j-\delta_i\notin\Delta_+$ for all i,j, and for $i<n$, we have $\varepsilon_j-\delta_i\in\Delta_+$ iff $j\leq i$. The assumption $wS\subset\Delta_+$ means that $w\varepsilon_i-\delta_i\in\Delta_+$ for all $i<n$ and $w\varepsilon_m+\delta_n\in\Delta_+$. For $i<n$, this gives $w\varepsilon_i=\varepsilon_{j_i}$ for some $j_i\leq i$. Hence, $w\varepsilon_i=\varepsilon_i$ for $i=1,\dots,n-1$. The remaining condition $w\varepsilon_m+\delta_n\in\Delta_+$ means that $w\varepsilon_m=\varepsilon_{j_m}$ or $w\varepsilon_m=-\varepsilon_m$.

For $m = n + 1$, we have $\rho = 0$, so $w \in A$ iff $w \in W^\#$, $wS \subset \Delta_+$. Thus, by the above, $w \in A$ iff $w\varepsilon_i = \varepsilon_i$ for $i < n = m - 1$ and $w\varepsilon_m \in \{\pm\varepsilon_m, \varepsilon_{m-1}\}$.

Take $m > n + 1$. The roots $\beta \in \Delta^\#_+$ such that $(\rho, \beta) = 0$ are of the form $\beta = \varepsilon_i - \varepsilon_j$ for $i < j \leq n$ or $\beta = \varepsilon_{m-1} \pm \varepsilon_m$. From Lemma 7.1.1 we conclude that the subgroup $\mathrm{Stab}_{W^\#}\,\rho$ is a product of S_n defined in 7.4.1 and the group generated by the reflections $s_{\varepsilon_{m-1}\pm\varepsilon_m}$. By the above, $w \in A$ iff $w\varepsilon_i = \varepsilon_i$ for $i < m - 1$ and $w\varepsilon_m \in \{\pm\varepsilon_m, \varepsilon_{m-1}\}$.

Since $W^\#$ is the Weyl group of $D(m)$, i.e. the group of signed permutations of $\{\varepsilon_i\}_{i=1}^m$, changing the even number of signs, the set

$$\left\{w \in W^\# \mid w\varepsilon_i = \varepsilon_i \text{ for } i < m - 1 \;\&\; w\varepsilon_m \in \{\pm\varepsilon_m, \varepsilon_{m-1}\}\right\}$$

is $\{\mathrm{id}, s_{\varepsilon_{m-1}-\varepsilon_m}, s_{\varepsilon_{m-1}-\varepsilon_m}s_{\varepsilon_{m-1}+\varepsilon_m}\}$. Hence $A = \{\mathrm{id}, s_{\varepsilon_{m-1}-\varepsilon_m}, s_{\varepsilon_{m-1}-\varepsilon_m}s_{\varepsilon_{m-1}+\varepsilon_m}\}$, as required. This establishes (ii) for the case $(\varepsilon_m + \delta_n) \in \Pi$.

7.6 Step (ii′) Consider the case $\mathfrak{gl}(n|n)$. We have $\rho = 0$. Set $\xi := \sum_{\beta\in S}\beta = \sum\varepsilon_i - \sum\delta_i$. Let us verify that

$$\mathbb{Q}\xi \cap \mathrm{supp}\big(Re^\rho - X\big) = \emptyset. \tag{9}$$

Indeed, it is easy see that ξ has a unique presentation as a positive linear combination of positive roots:

$$\xi = \sum_{\alpha\in\Delta_+} m_\alpha\alpha, \quad m_\alpha \geq 0 \Longrightarrow m_\beta = 1 \text{ for } \beta \in S,\; m_\alpha = 0 \text{ for } \alpha \notin S. \tag{10}$$

This implies that for $s \notin \mathbb{Z}_{\geq 0}$, the coefficients of $e^{-s\xi}$ in $Re^\rho = R$ and in X are equal to zero, and that the coefficient of $e^{-s\xi}$ in R is equal to $(-1)^{sn}$ for $s \in \mathbb{Z}_{\geq 0}$. It remains to show that the coefficient of $e^{-s\xi}$ in X is $(-1)^{sn} = (-1)^{\mathrm{ht}(s\xi)}$. It is enough to verify that $|w|\mu - \varphi(w) = s\xi$ for $w \in W^\#$ implies $w = \mathrm{id}$. Assume that $|w|\mu - \varphi(w) = s\xi$. By definition, $|w|\mu, -\varphi(w) \in Q^+$. From (10) we conclude that $|w|\mu, -\varphi(w) \in \mathbb{Z}_{\geq 0}S$. Recall that $\varphi(w) = \sum_{\beta\in S: w\beta\in\Delta_-} w\beta$. By (10), $-\varphi(w) \in \mathbb{Z}_{\geq 0}S$ implies $(-w\beta) \in S$ for each $\beta \in S$ such that $w\beta \in \Delta_-$. However, $(-w\beta) \notin S$ for any $w \in W^\#$ and $\beta \in S$, because $-w(\varepsilon_i - \delta_i) = \delta_i - w\varepsilon_i \notin S$. Thus $wS \subset \Delta_+$, that is $w\varepsilon_i = \varepsilon_{i_j}$ for $i_j \leq i$. Hence $w = \mathrm{id}$. This establishes (9).

8 *W*-Invariance: Step (iii)

In this section we prove that X defined by the formula (2) is W-skew-invariant for a certain admissible pair (S, Π); for the case $D(m, n)$, $m > n$, we prove this for two admissible pairs (S, Π) and (S', Π') which are representatives of the equivalence classes defined in Sect. 6.

Recall that X is $W^\#$-skew-invariant and that $\Delta = \Delta^\# \sqcup \Delta_2$, that is $W = W^\# \times W_2$.

8.1 Operator F Recall the operator $F : \mathcal{R} \to \mathcal{R}$ given by the formula $F(Y) := \sum_{w \in W^{\#}} \operatorname{sgn}(w) wY$. Clearly, $w(F(Y)) = F(wY)$ for $w \in W_2$ and $F(wY) = w(F(Y)) = \operatorname{sgn}(w)F(Y)$ for $w \in W^{\#}$. In particular, $F(Y) = 0$ if $wY = Y$ for some $w \in W^{\#}$ with $\operatorname{sgn}(w) = -1$. We have

$$X = F(Y), \quad \text{where } Y := \frac{e^{\rho}}{\prod_{\beta \in S}(1 + e^{-\beta})}.$$

Suppose that $B \in \mathcal{R}$ is such that $w_2 w_1 B = B$ for some $w_1 \in W^{\#}$, $w_2 \in W_2$, where $\operatorname{sgn}(w_1 w_2) = 1$. Then

$$w_2^{-1} F(B) = F\big(w_2^{-1} B\big) = F(w_1 B) = \operatorname{sgn}(w_1) F(B),$$

that is $w_2 F(B) = \operatorname{sgn}(w_2) F(B)$.

As a result, in order to verify W-skew-invariance of $F(B)$ for an arbitrary $B \in \mathcal{R}$, it is enough to show that for each generator y of W_2, there exists $z \in W^{\#}$ such that $\operatorname{sgn}(yz) = 1$ and $yzB = B$ (we consider y running through a set of generators of W_2).

8.2 Root Systems We retain the notation of 7.2 for $\Delta_{0,+}$ and Δ_1, but, except for $A(m-1, n-1)$, we do not choose the same pairs (S, Π) as in 7.3.

For $A(m-1, n-1)$, we choose S, Π as in 7.3 ($S := \{\varepsilon_i - \delta_i\}$). For other cases, we choose $S := \{\delta_{n-i} - \varepsilon_{m-i}\}_{i=0}^{n-1}$ and

$$\begin{aligned} \Pi &:= P \cup \{\varepsilon_m\} && \text{for } B(m,n),\ B(n,m), \\ \Pi &:= P \cup \{\varepsilon_m + \delta_n\} && \text{for } D(m,n),\ m > n, \\ \Pi &:= P \cup \{2\varepsilon_m\} && \text{for } D(n,m),\ m \geq n, \end{aligned}$$

where $P := \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-n-1} - \varepsilon_{m-n}, \varepsilon_{m-n} - \delta_1, \delta_1 - \varepsilon_{m-n+1}, \varepsilon_{m-n+1} - \delta_2, \ldots, \delta_n - \varepsilon_m\}$ for $m > n$, and $P := \{\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \ldots, \delta_n - \varepsilon_n\}$ for $m = n$. For the case of $D(m,n)$, $m > n$, we consider two admissible pairs (S, Π) and (S', Π), where $S' := \{\delta_{n-i} - \varepsilon_{m-i}\}_{i=1}^{n-1} \cup \{\delta_n + \varepsilon_m\}$.

Recall that $(\varepsilon_i, \varepsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$ and notice that $(\alpha, \alpha) \geq 0$ for all $\alpha \in \Pi$.

8.3 S_n-Invariance Let $S_n \subset W_2$ be the group of permutations of $\delta_1, \ldots, \delta_n$. In all cases, S is of the form $S = \pm\{\delta_i - \varepsilon_{r+i}\}$ for $r = 0$ or $r = m - n$. For $i = 1, \ldots, n-1$, we have $(\rho, \delta_i - \delta_{i+1}) = (\rho, \varepsilon_{r+i} - \varepsilon_{r+i+1}) = 0$. Therefore the reflections $s_{\varepsilon_{r+i} - \varepsilon_{r+i+1}}, s_{\delta_i - \delta_{i+1}}$ stabilize ρ. Since $s_{\varepsilon_{r+i} - \varepsilon_{r+i+1}} s_{\delta_i - \delta_{i+1}}$ stabilizes the elements of S, we have $s_{\varepsilon_{r+i} - \varepsilon_{r+i+1}} s_{\delta_i - \delta_{i+1}} Y = Y$ for $i = 1, \ldots, n-1$. Using 8.1, we conclude that X is S_n-skew-invariant. In particular, this establishes W-invariance of X for $A(m,n)$-case.

For the case $D(m,n)$, $m > n$, consider the admissible pair (S', Π). Arguing as above, we see that $s_{\delta_i - \delta_{i+1}}(X) = -X$ for $i = 1, \ldots, n-2$. Since $\delta_{n-1} - \varepsilon_{m-1}, \delta_n + \varepsilon_m \in S'$, the product $w := s_{\varepsilon_{m-1} + \varepsilon_m} s_{\delta_{n-1} - \delta_n}$ stabilizes the elements of S'. Since $(\rho, \delta_i) = (\rho, \varepsilon_{m-n+i}) = 0$ for $i = 1, \ldots, n$, w stabilizes ρ. Thus, $wY = Y$, and so $s_{\delta_{n-1} - \delta_n} X = -X$ by 8.1. Hence, X is S_n-skew-invariant.

8.4 Cases $B(m,n)B$, $B(n,m)$, $D(m,n)$, $m>n$ In this case, W_2 is the group of signed permutations of $\{\delta_i\}_{i=1}^n$, so it is generated by s_{δ_n} and the elements of S_n. In the light of 8.1 and 8.3, it is enough to verify that $s_{\delta_n}s_{\varepsilon_m}Y=Y$. Set

$$\beta:=\delta_n-\varepsilon_m\in S.$$

Consider the cases $B(m,n)$, $B(n,m)$. In this case, $W^\#$ is the group of signed permutations of $\{\varepsilon_i\}_{i=1}^m$. Since $\beta,\varepsilon_m\in\Pi$, we have $(\rho,\varepsilon_m)=(\rho,\delta_n)=\frac{1}{2}$, so $s_{\delta_n}s_{\varepsilon_m}\rho=\rho+\beta$. Clearly, $s_{\delta_n}s_{\varepsilon_m}$ stabilizes the elements of $S\setminus\{\beta\}$, and $s_{\delta_n}s_{\varepsilon_m}\beta=-\beta$. As a result, $s_{\delta_n}s_{\varepsilon_m}Y=Y$, as required.

Consider the case $D(m,n)$, $m>n$. In this case, $W^\#$ is the group of signed permutations of $\{\varepsilon_i\}$ that change an even number of signs. Notice that $s_{\varepsilon_{m-n}}s_{\varepsilon_m}\in W^\#$ and $\operatorname{sgn}(s_{\varepsilon_{m-n}}s_{\varepsilon_m})=1$. Set $w:=s_{\varepsilon_{m-n}}s_{\varepsilon_m}s_{\delta_n}$. We have $(\rho,\delta_n)=(\rho,\varepsilon_m)=(\rho,\varepsilon_{m-n})=0$, so $w\rho=\rho$. Since w stabilizes the elements of $S\setminus\{\beta\}$, we have $wY=e^{-\beta}Y$. We obtain

$$s_{\delta_n}F(Y)=s_{\delta_n}F(s_{\varepsilon_{m-n}}s_{\varepsilon_m}Y)=F(s_{\delta_n}s_{\varepsilon_{m-n}}s_{\varepsilon_m}Y)=F(wY)=F\big(e^{-\beta}Y\big),$$

and so

$$(1+s_{\delta_n})F(Y)=F\big((1+e^{-\beta})Y\big)=F\left(\frac{e^\rho}{\prod_{\beta'\in S\setminus\{\beta\}}(1+e^{-\beta'})}\right)=0,$$

where the last equality follows from the fact that the reflection $s_{\varepsilon_{m-n}-\varepsilon_m}$ stabilizes ρ and $S\setminus\{\beta\}$. Hence $(1+s_{\delta_n})F(Y)=0$, as required.

For the admissible pair (S',Π), we obtain the required formula $(1+s_{\delta_n})F(Y)=0$ along the same lines substituting β by $\delta_n+\varepsilon_m$.

7.5 Case $D(n,m)$, $m\geq n$ In this case, $W^\#$ is the group of signed permutations of $\{\varepsilon_i\}$, and W_2 is the group of signed permutations of $\{\delta_i\}_{i=1}^n$ that change an even number of signs. Note that the reflection s_{δ_i} does not lie in W_2, but $s_{\delta_i}\Delta=\Delta$, so s_{δ_i} acts on $\mathcal{R}$, and this action commutes with the operator F. Since W_2 is generated by $s_{\delta_1}s_{\delta_2}$ and the elements of S_n, it is enough to verify that $s_{\delta_1}s_{\delta_2}F(Y)=F(Y)$. Set $\beta_i:=\delta_i-\varepsilon_{m-n+i}\in S$. We have $(\rho,\varepsilon_{m-n+i})=(\rho,\delta_i)=1$, so $s_{\delta_i}s_{\varepsilon_m+n-i}\rho=\rho+2\beta_i$, that is $s_{\delta_i}s_{\varepsilon_m+n-i}Y=e^{\beta_i}Y$. Therefore,

$$\begin{aligned}(1-s_{\delta_i})F(Y)&=F(Y)+F(s_{\delta_i}s_{\varepsilon_m+n-i}Y)=F\big((1+e^{\beta_i})Y\big)\\&=F\left(\frac{e^{\rho+\beta_i}}{\prod_{\beta\in S\setminus\{\beta_i\}}(1+e^{-\beta})}\right)=0,\end{aligned}$$

because $s_{\varepsilon_m+n-i}\in W^\#$ stabilizes $\rho+\beta_i$ and the elements of $S\setminus\{\beta_i\}$. Thus, $s_{\delta_i}F(Y)=F(Y)$, and so $s_{\delta_i}s_{\delta_j}F(Y)=F(Y)$ for any i,j. Hence, $X=F(Y)$ is W_2-skew-invariant.

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Hopf Algebras and Frobenius Algebras in Finite Tensor Categories

Christoph Schweigert and Jürgen Fuchs

Abstract We discuss algebraic and representation theoretic structures in braided tensor categories $\mathcal{C}$ which obey certain finiteness conditions. A lot of interesting structure of such a category is encoded in a Hopf algebra $\mathcal{H}$ in $\mathcal{C}$. In particular, the Hopf algebra $\mathcal{H}$ gives rise to representations of the modular group $\mathrm{SL}(2,\mathbb{Z})$ on various morphism spaces. We also explain how every symmetric special Frobenius algebra in a semisimple modular category provides additional structure related to these representations.

Keywords Finite tensor category · Hopf algebra · Modular group · Frobenius algebra

Mathematics Subject Classification (2010) 16Txx · 16B50 · 18D10

1 Braided Finite Tensor Categories

Algebra and representation theory in semisimple ribbon categories has been an active field over the last decade, having applications to quantum groups, low-dimensional topology and quantum field theory. More recently, partly in connection with progress in the understanding of logarithmic conformal field theories, there has been increased interest in tensor categories that are not semisimple any longer, but still obey certain finiteness conditions [EO].

Owing to the work of various groups (for some recent results, see e.g. [GT, NT]), examples of such categories are by now rather explicitly understood, at least as abelian categories. In this section we describe a class of categories that has received particular attention. This will allow us to define the structure of a semisimple modular tensor category. To extend the notion of modular tensor category to the non-semisimple case requires further categorical constructions involving Hopf algebras

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and coends; these will be introduced in Sect. 2. These constructions also provide representations of the modular group $\mathrm{SL}(2,\mathbb{Z})$ on certain morphism spaces. In Sect. 3 we show that symmetric special Frobenius algebras in semisimple modular tensor categories give rise to structures related to such $\mathrm{SL}(2,\mathbb{Z})$-representations.

Let $\Bbbk$ be an algebraically closed field of characteristic zero, and $\mathcal{V}ect_{\mathrm{fin}}(\Bbbk)$ the category of finite-dimensional $\Bbbk$-vector spaces.

Definition 1.1 A *finite category* $\mathcal{C}$ is an abelian category enriched over $\mathcal{V}ect_{\mathrm{fin}}(\Bbbk)$ with the following additional properties:

1. Every object has finite length.
2. Every object $X \in \mathcal{C}$ has a projective cover $P(X) \in \mathcal{C}$.
3. The set I of isomorphism classes of simple objects is finite.

It can be shown that an abelian category is a finite category if and only if it is equivalent to the category of (left, say) modules over a finite-dimensional $\Bbbk$-algebra.

We will be concerned with finite categories that have additional structure. First, they are tensor categories, i.e., for our purposes, sovereign monoidal categories:

Definition 1.2 A *tensor category* over a field $\Bbbk$ is a $\Bbbk$-linear abelian monoidal category $\mathcal{C}$ with simple tensor unit $\mathbf{1}$ and with both a left and a right duality in the sense of [Ka, Definition XIV.2.1] such that the category is sovereign, i.e. the two functors

$$?^\vee, {}^\vee? : \mathcal{C} \to \mathcal{C}^{\mathrm{opp}}$$

that are induced by the left and right dualities coincide.

Thus, for any object $V \in \mathcal{C}$, there exists an object $V^\vee = {}^\vee V \in \mathcal{C}$ together with morphisms

$$b_V : \mathbf{1} \to V \otimes V^\vee \quad \text{and} \quad d_V : V^\vee \otimes V \to \mathbf{1}$$

(right duality) and

$$\tilde{b}_V : \mathbf{1} \to V^\vee \otimes V \quad \text{and} \quad \tilde{d}_V : V \otimes V^\vee \to \mathbf{1}$$

(left duality), obeying the relations

$$(\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V \quad \text{and} \quad (d_V \otimes \mathrm{id}_{V^\vee}) \circ (\mathrm{id}_{V^\vee} \otimes b_V) = \mathrm{id}_{V^\vee}$$

and analogous relations for the left duality, and the duality functors not only coincide on objects, but also on morphisms, i.e.

$$\begin{aligned} &(d_V \otimes \mathrm{id}_{U^\vee}) \circ (\mathrm{id}_{V^\vee} \otimes f \otimes \mathrm{id}_{U^\vee}) \circ (\mathrm{id}_{V^\vee} \otimes b_U) \\ &\quad = (\mathrm{id}_{U^\vee} \otimes \tilde{d}_V) \circ (\mathrm{id}_{U^\vee} \otimes f \otimes \mathrm{id}_{V^\vee}) \circ (\tilde{b}_U \otimes \mathrm{id}_{V^\vee}) \end{aligned}$$

for all morphisms $f : U \to V$.

To give an example, the category of finite-dimensional left modules over any finite-dimensional complex Hopf algebra H is a finite tensor category. As a direct consequence of the definition, the tensor product functor $\otimes$ is exact in both arguments. We will impose on the dualities the additional requirement that left and right dualities lead to the same cyclic trace $\mathrm{tr}\colon \mathrm{End}(U) \to \mathrm{End}(\mathbf{1})$ and thus to the same dimension $\dim(U) = \mathrm{tr}(\mathrm{id}_U)$.

The categories of our interest have in addition a braiding:

Definition 1.3 A *braiding* on a tensor category $\mathcal{C}$ is a natural isomorphism

$$c : \otimes \to \otimes^{\mathrm{opp}}$$

that is compatible with the tensor product, i.e. satisfies

$$c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}) \quad \text{and}$$
$$c_{U,V\otimes W} = (id_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W).$$

We choose a set $\{U_i\}_{i\in I}$ of representatives for the isomorphism classes of simple objects and take the tensor unit to be the representative of its isomorphism class, writing $\mathbf{1} = U_0$.

We are now ready to formulate the notion of a modular tensor category. Our definition will, however, still be preliminary, as it has the disadvantage of being sensible only for semisimple categories.

Definition 1.4 A semisimple *modular tensor category* is a semisimple finite braided tensor category such that the matrix $(S_{ij})_{i,j\in I}$ with entries

$$S_{ij} := \mathrm{tr}(c_{U_j,U_i} \circ c_{U_i,U_j})$$

is non-degenerate.

Two remarks are in order:

Remarks 1.5

1. The representation categories of several algebraic structures give examples of semisimple modular tensor categories:
   (a) Left modules over connected factorizable ribbon weak Hopf algebras with Haar integral over an algebraically closed field [NTV].
   (b) Local sectors of a finite μ-index net of von Neumann algebras on $\mathbb{R}$ if the net is strongly additive and split [KLM].
   (c) Representations of selfdual C_2-cofinite vertex algebras with an additional finiteness condition on the homogeneous components and which have semisimple representation categories [Hu].

2. By the results of Reshetikhin and Turaev [RT, T], every $\mathbb{C}$-linear semisimple modular tensor category $\mathcal{C}$ provides a three-dimensional topological field theory, i.e. a tensor functor

$$\mathrm{tft}_{\mathcal{C}} : \mathrm{cobord}_{3,2}^{\mathcal{C}} \to \mathcal{V}ect_{\mathrm{fin}}(\mathbb{C}).$$

Here $\mathrm{cobord}_{3,2}^{\mathcal{C}}$ is a category of three-dimensional cobordisms with embedded ribbon graphs that are decorated by objects and morphisms of $\mathcal{C}$.

There are also various results for the case of non-semisimple modular categories. We refer to [He, L1, V] for the construction of three-manifold invariants, to [L1] for the construction of representations of mapping class groups, and to [KL] for an attempt to unify these constructions in terms of a topological quantum field theory defined on a double category of manifolds with corners.

2 Hopf Algebras, Coends and Modular Tensor Categories

Our goal is to study some algebraic and representation theoretic structures in tensor categories of the type introduced above. To simplify the exposition, we suppose that we have replaced the tensor category $\mathcal{C}$ by an equivalent strict tensor category. For a review of tensor categories and related notions, we refer to [BK].

Definition 2.1 A (unital, associative) *algebra* in a (strict) tensor category $\mathcal{C}$ is a triple consisting of an object $A \in \mathcal{C}$, a multiplication morphism $m \in \mathrm{Hom}(A \otimes A, A)$ and a unit morphism $\eta \in \mathrm{Hom}(\mathbf{1}, A)$, subject to the relations

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m) \quad \text{and} \quad m \circ (\eta \otimes \mathrm{id}_A) = \mathrm{id}_A = m \circ (\mathrm{id}_A \otimes \eta)$$

which express associativity and unitality.

Analogously, a *coalgebra* in $\mathcal{C}$ is a triple consisting of an object C, a comultiplication $\Delta\colon\ C \to C \otimes C$ and a counit $\varepsilon\colon\ C \to \mathbf{1}$ obeying coassociativity and counit conditions.

Similarly one generalizes other basic notions of algebra to the categorical setting and introduces modules, bimodules, comodules, etc. (For a more complete exposition, we refer to [FRS1].)

To proceed, we observe that the multiplication of an algebra A endows both A itself and $A \otimes A$ with the structure of an A-bimodule. Further, if the category $\mathcal{C}$ is braided, then the object $A \otimes A$ can be endowed with the structure of a unital associative algebra by taking the morphisms $(m \otimes m) \circ (\mathrm{id}_A \otimes c_{A,A} \otimes \mathrm{id}_A)$ as the product and $\eta \otimes \eta$ as the counit.

Definition 2.2 Let $\mathcal{C}$ be a tensor category, and $A \in \mathcal{C}$ an object which is endowed with both the structure (A, m, η) of a unital associative algebra and the structure (A, Δ, ε) of a counital coassociative coalgebra.

1. $(A, m, \eta, \Delta, \varepsilon)$ is called a *Frobenius algebra* iff $\Delta\colon\ A \to A \otimes A$ is a morphism of bimodules.

2. $(A, m, \eta, \Delta, \varepsilon)$ is called a *bialgebra* iff $\Delta\colon A \to A \otimes A$ is a morphism of unital algebras.
3. A bialgebra with an antipode $\mathrm{S}\colon A \to A$ (with properties analogous to the classical case) is called a *Hopf algebra*.

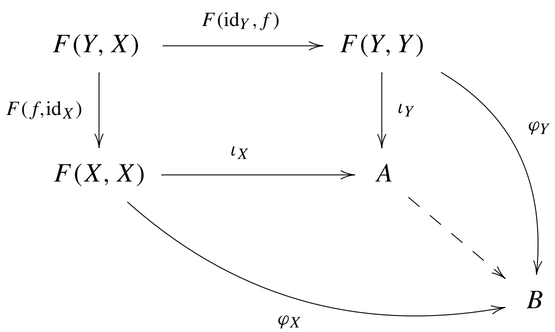
To construct concrete examples of such structures, we recall a few notions from category theory.

Definition 2.3 Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F\colon \mathcal{C}^{\mathrm{opp}} \times \mathcal{C} \to \mathcal{D}$ be a functor.

1. For B an object of $\mathcal{D}$, a *dinatural transformation* $\varphi\colon F \Rightarrow B$ is a family of morphisms $\varphi_X\colon F(X, X) \to B$ for every object $X \in \mathcal{C}$ such that the diagram

$$\begin{array}{ccc} F(Y,X) & \xrightarrow{F(\mathrm{id}_Y, f)} & F(Y,Y) \\ {\scriptstyle F(f,\mathrm{id}_X)}\downarrow & & \downarrow{\scriptstyle \varphi_Y} \\ F(X,X) & \xrightarrow{\varphi_X} & B \end{array}$$

commutes for all morphisms $X \xrightarrow{f} Y$ in $\mathcal{C}$.
2. A *coend* for the functor F is a dinatural transformation $\iota\colon F \Rightarrow A$ with the universal property that any dinatural transformation $\varphi\colon F \Rightarrow B$ uniquely factorizes:



If the coend exists, it is unique up to unique isomorphism. It is denoted by $\int^X F(X, X)$. The universal property implies that a morphism with domain $\int^X F(X, X)$ can be specified by a dinatural family of morphisms $F(X, X) \to B$ for each object $X \in \mathcal{C}$.

We are now ready to formulate the following result.

Theorem 2.4 [L2] *In a finite braided tensor category* $\mathcal{C}$, *the coend*

$$\mathcal{H} := \int^X X^\vee \otimes X$$

of the functor

$$F : \mathcal{C}^{\mathrm{opp}} \times \mathcal{C} \to \mathcal{C}$$
$$(U, V) \mapsto U^\vee \otimes V$$

exists, and it has a natural structure of a Hopf algebra in $\mathcal{C}$.

Proof For a proof, we refer e.g. to [V]. Here we only indicate how the structural morphisms of the Hopf algebra are constructed. Owing to the universal property, the counit $\varepsilon_{\mathcal{H}}\colon\ \mathcal{H} \to \mathbf{1}$ can be specified by the dinatural family

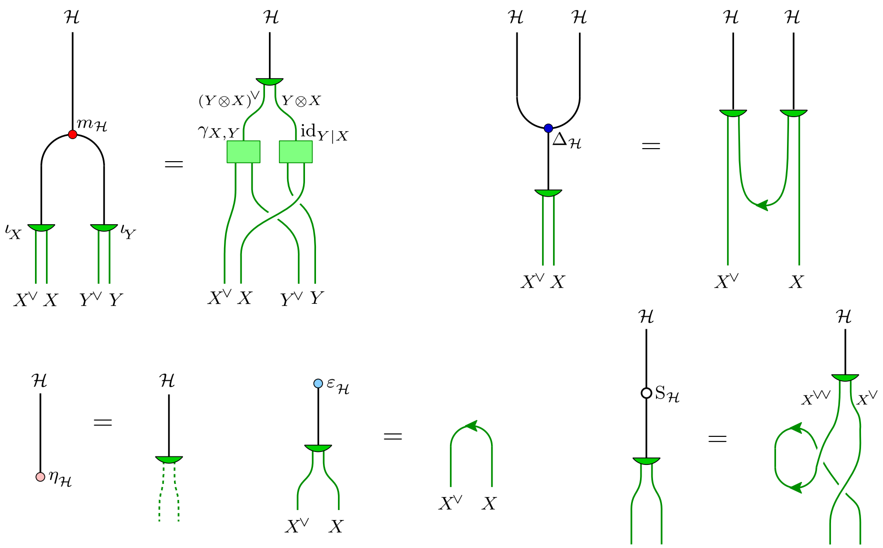
$$\varepsilon_{\mathcal{H}} \circ \iota_X = d_X : X^\vee \otimes X \to \mathbf{1}$$

of morphisms. Similarly, the coproduct is given by the dinatural family

$$\Delta_{\mathcal{H}} \circ \iota_{\mathcal{H}} = (\iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^\vee} \otimes b_X \otimes \mathrm{id}_X) : X^\vee \otimes X \to \mathcal{H} \otimes \mathcal{H}.$$

It should be appreciated that the braiding does not enter in the coalgebra structure of $\mathcal{H}$.

It does enter in the product, though. We refrain from writing out the product as a formula. Instead, we use the graphical formalism [JS, FRS1] to display all structural morphisms ($m_{\mathcal{H}}, \Delta_{\mathcal{H}}, \eta_{\mathcal{H}}, \epsilon_{\mathcal{H}}, \mathrm{S}_{\mathcal{H}}$) of the Hopf algebra $\mathcal{H}$. More precisely, we display dinatural families of morphisms so that the identities apply to all $X, Y \in \mathcal{C}$:



(Here $\gamma_{X,Y}$ is the canonical identification of $X^\vee \otimes Y^\vee$ with $(Y \otimes X)^\vee$, and $\mathrm{id}_{X|Y}$ is the one of $\mathrm{id}_X \otimes \mathrm{id}_Y$ with $\mathrm{id}_{X\otimes Y}$.) □

An explicit description of the Hopf algebra $\mathcal{H} \in \mathcal{C}$ is available in the following specific situations:

Examples 2.5

1. For $\mathcal{C} = H$-mod the category of left modules over a finite-dimensional ribbon Hopf algebra H, the coend $\mathcal{H} = \int^X X^\vee \otimes X$ is the dual space $H^* = \mathrm{Hom}_\Bbbk(H, \Bbbk)$ endowed with the coadjoint representation. The structure morphism for the coend for a module $M \in H$-mod is

$$\iota_M : M^\vee \otimes M \to H^*$$
$$\tilde{m} \otimes m \mapsto \big(h \mapsto \langle \tilde{m}, h.m\rangle\big).$$

   For more details, see [V, Sect. 4.5].
2. If the finite tensor category $\mathcal{C}$ is semisimple, then the Hopf algebra decomposes as an object as $\mathcal{H} = \bigoplus_{i\in I} U_i^\vee \otimes U_i$, see [V, Sect. 3.2].

The Hopf algebra in question has additional structure: it comes with an integral and with a Hopf pairing.

Definition 2.6 A *left integral* of a bialgebra $(H, m, \eta, \Delta, \varepsilon)$ in $\mathcal{C}$ is a non-zero morphism $\mu_l \in \mathrm{Hom}(\mathbf{1}, H)$ satisfying

$$m \circ (\mathrm{id}_H \otimes \mu_l) = \mu_l \circ \varepsilon.$$

A *right cointegral* of H is a non-zero morphism $\lambda_r \in \mathrm{Hom}(H, \mathbf{1})$ satisfying

$$(\lambda \otimes \mathrm{id}_H) \circ \Delta = \eta \circ \lambda.$$

Right integrals μ_r and left cointegrals λ_l are defined analogously.

The Hopf algebra $\mathcal{H}$ in any finite braided tensor category has left and right integrals, as can be shown [L2] by a generalization of the classical argument of Sweedler that an integral exists for any finite-dimensional Hopf algebra. If $\mathcal{C}$ is semisimple, then the integral of $\mathcal{H}$ can be given explicitly [Ke, Sect. 2.5]:

$$\mu_l = \mu_r = \bigoplus_{i\in I} \dim(U_i) b_{U_i}.$$

Remarks 2.7

1. If the left and right integrals of $\mathcal{H}$ coincide, then the integral can be used as a Kirby element and provides invariants of three-manifolds [V]. If the category $\mathcal{C}$ is the category of representations of a finite-dimensional Hopf algebra, this is the Hennings–Lyubashenko [L1] invariant.
2. The category $\mathcal{C}$ is semisimple if and only if the morphism $\varepsilon \circ \mu \in \mathrm{Hom}(\mathbf{1}, \mathbf{1})$ does not vanish, i.e. iff the constant $\mathcal{D}^2$ of proportionality in

$$\varepsilon \circ \mu = \mathcal{D}^2\, \mathrm{id}_\mathbf{1}$$

is non-zero. (This generalizes Maschke's theorem.) This constant, in turn, which in the semisimple case (with $\mu_l = \mu_r$ normalized as above) has the value $\mathcal{D}^2 = \sum_{i \in I} (\dim U_i)^2$, crucially enters the normalizations in the Reshetikhin–Turaev construction of topological field theories (see e.g. Chap. II of [T]).

Invariants based on nonsemisimple categories, like the Hennings invariant, vanish on many three-manifolds. This can be traced back to the vanishing of $\varepsilon \circ \mu$ [CKS].

3. Any Hopf algebra H in $\mathcal{C}$ with invertible antipode that has a left integral μ and a right cointegral λ with $\lambda \circ \mu \neq 0$ is naturally also a Frobenius algebra, with the same algebra structure.

Definition 2.8 A *Hopf pairing* of a Hopf algebra H in $\mathcal{C}$ is a morphism

$$\omega_H : H \otimes H \to \mathbf{1}$$

such that

$$\omega_H \circ (m \otimes \mathrm{id}_H) = (\omega_H \otimes \omega_H) \circ (\mathrm{id}_H \otimes c_{H,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \mathrm{id}_H \otimes \Delta),$$

$$\omega_H \circ (\mathrm{id}_H \otimes m) = (\omega_H \otimes \omega_H) \circ \left(\mathrm{id}_H \otimes c_{H,H}^{-1} \otimes \mathrm{id}_H\right) \circ (\Delta \otimes \mathrm{id}_H \otimes \mathrm{id}_H)$$

and

$$\omega_H \circ (\eta \otimes \mathrm{id}_H) = \varepsilon = \omega_H \circ (\mathrm{id}_H \otimes \eta).$$

As one easily checks, a non-degenerate Hopf pairing gives an isomorphism $H \to H^\vee$ of Hopf algebras.

The dinatural family of morphisms

$$(d_X \otimes d_Y) \circ \left[\mathrm{id}_{X^\vee} \otimes (c_{Y^\vee,X} \circ c_{X,Y^\vee}) \otimes \mathrm{id}_Y\right]$$

induces a bilinear pairing $\omega_{\mathcal{H}}\colon\; \mathcal{H} \otimes \mathcal{H} \to \mathbf{1}$ on the coend $\mathcal{H} = \int^X X^\vee \otimes X$ of a finite braided tensor category. It endows [L1] the Hopf algebra $\mathcal{H}$ with a symmetric Hopf pairing.

We are now finally in a position to give a conceptual definition of a modular finite tensor category without requiring it to be semisimple:

Definition 2.9 [KL, Definition 5.2.7] A *modular finite tensor category* is a braided finite tensor category for which the Hopf pairing $\omega_{\mathcal{H}}$ is non-degenerate.

Example 2.10 The category H-mod of left modules over a finite-dimensional factorizable ribbon Hopf algebra H is a modular finite tensor category [LM, L1].

One can show [L2, Theorem 6.11] that if $\mathcal{C}$ is modular in the sense of Definition 2.9, then the left integral and the right integral of $\mathcal{H}$ coincide.

As the terminology suggests, there is a relation with the modular group $\mathrm{SL}(2,\mathbb{Z})$. To see this, we will now obtain elements $S_{\mathcal{H}}, T_{\mathcal{H}} \in \mathrm{End}(\mathcal{H})$ that satisfy the relations for generators of $\mathrm{SL}(2,\mathbb{Z})$.

Recall the notion of the center $Z(\mathcal{C})$ of a category as the algebra of natural endotransformations of the identity endofunctor of $\mathcal{C}$ [Ma]. Given such a natural transformation $(\phi_X)_{X\in\mathcal{C}}$ with $\phi_X \in \mathrm{End}(X)$, one checks that $(\iota_X \circ (\mathrm{id}_{X^\vee} \otimes \phi_X))_{X\in\mathcal{C}}$ is a dinatural family, so that the universal property of the coend gives us a unique endomorphism $\overline{\phi}_{\mathcal{H}}$ of $\mathcal{H}$ such that the diagram

$$\begin{array}{ccc} X^\vee \otimes X & \xrightarrow{\mathrm{id}\otimes\phi_X} & X^\vee \otimes X \\ {\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle \iota_X} \\ \mathcal{H} & \underset{\overline{\phi}_{\mathcal{H}}}{\dashrightarrow} & \mathcal{H} \end{array}$$

commutes, leading to an injective linear map $Z(\mathcal{C}) \to \mathrm{End}(\mathcal{H})$.

Since $\mathcal{H}$ has in particular the structure of a coalgebra and $\mathbf{1}$ the structure of an algebra, the vector space $\mathrm{Hom}(\mathcal{H},\mathbf{1})$ has a natural structure of a $\Bbbk$-algebra. Concatenating with the counit $\varepsilon_{\mathcal{H}}$ gives a map

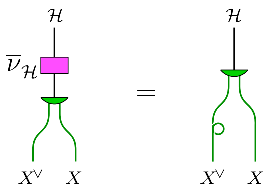
$$Z(\mathcal{C}) \longrightarrow \mathrm{End}(\mathcal{H}) \xrightarrow{(\varepsilon_{\mathcal{H}})_*} \mathrm{Hom}(\mathcal{H},\mathbf{1}),$$

which can be shown [Ke, Lemma 4] to be an isomorphism of $\Bbbk$-algebras. The vector space on the right-hand side is dual to the vector space $\mathrm{Hom}(\mathbf{1},\mathcal{H})$, of which one can think as the appropriate substitute for the space of class functions. Hence $\mathrm{Hom}(\mathbf{1},\mathcal{H})$ would be a natural starting point for constructing a vector space assigned to the torus T^2 by a topological field theory based on $\mathcal{C}$.

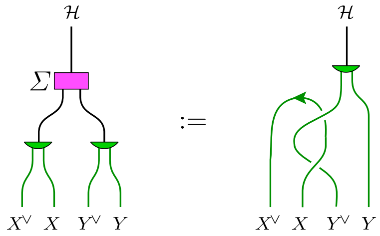
If the category $\mathcal{C}$ is a ribbon category, we have the ribbon element $\nu \in Z(\mathcal{C})$. We set

$$T_{\mathcal{H}} := \overline{\nu}_{\mathcal{H}} \in \mathrm{End}(\mathcal{H}).$$

Pictorially,



Another morphism $\Sigma\colon \mathcal{H}\otimes\mathcal{H}\to\mathcal{H}$ is obtained from the following family of morphisms which is dinatural both in X and in Y:



Composing this morphism to $\mathcal{H}$ with a left or right integral $\mu\colon \mathbf{1}\to\mathcal{H}$, we arrive at an endomorphism

$$S_{\mathcal{H}} := \Sigma\circ(\mathrm{id}_{\mathcal{H}}\otimes\mu)\in\mathrm{End}(\mathcal{H}).$$

For $\xi\in\Bbbk^\times$, denote by $\Bbbk_\xi\mathrm{SL}(2,\mathbb{Z})$ the twisted group algebra of $\mathrm{SL}(2,\mathbb{Z})$ with relations $S^4=1$ and $(ST)^3=\xi S^2$. The previous construction and the following result are due to Lyubashenko.

Theorem 2.11 [L2, Sect. 6] *Let $\mathcal{C}$ be modular. Then the two-sided integral of $\mathcal{H}$ can be normalized in such a way that the endomorphisms $S_{\mathcal{H}}$ and $T_{\mathcal{H}}$ of $\mathcal{H}$ provide a morphism of algebras*

$$\Bbbk_\xi\mathrm{SL}(2,\mathbb{Z})\longrightarrow\mathrm{End}(\mathcal{H})$$

for some $\xi\in\Bbbk^\times$.

Since for every $U\in\mathcal{C}$, the morphism space $\mathrm{Hom}(U,\mathcal{H})$ is, by push-forward, a left module over the algebra $\mathrm{End}(\mathcal{H})$, we obtain this way projective representations of $\mathrm{SL}(2,\mathbb{Z})$ on all vector spaces $\mathrm{Hom}(U,\mathcal{H})$.

To set the stage for the results in the next section, we consider the map

$$\mathrm{Obj}(\mathcal{C})\to\mathrm{Hom}(\mathbf{1},\mathcal{H})$$

$$U\mapsto\chi_U$$

with

$$\chi_U:\mathbf{1}\xrightarrow{b_U}U^\vee\otimes U\xrightarrow{\iota_U}\mathcal{H}.$$

It factorizes to a morphism of rings

$$K_0(\mathcal{C})\to\mathrm{Hom}(\mathbf{1},\mathcal{H})=\mathrm{tft}_{\mathcal{C}}\big(T^2\big).$$

If the category $\mathcal{C}$ is semisimple, then $\mathrm{Hom}(\mathbf{1},\mathcal{H})\cong\bigoplus_{i\in I}\mathrm{Hom}(\mathbf{1},U_i^\vee\otimes U_i)$, so that $\{\chi_{U_i}\}_{i\in I}$ constitutes a basis of the vector space $\mathrm{Hom}(\mathbf{1},\mathcal{H})$. If $\mathcal{C}$ is not semisimple, these elements are still linearly independent, but they do not form a basis any more. Pseudo-characters [Mi, GT] have been proposed as a (non-canonical) complement of this linearly independent set.

3 Frobenius Algebras and Braided Induction

In this section we show that symmetric special Frobenius algebras (i.e. Frobenius algebras with two further properties, to be defined below) in a modular tensor category allow us to specify interesting structure related to the $\mathrm{SL}(2, \mathbb{Z})$-representation that we have just explained.

Given an algebra A in a braided (strict) tensor category, we consider the two tensor functors

$$\alpha_A^{\pm} : \mathcal{C} \to A\text{-bimod}$$
$$U \mapsto \alpha_A^{\pm}(U)$$

which assign to an object $U \in \mathcal{C}$ the bimodule $(A \otimes U, \rho_l, \rho_r)$ for which the left action is given by multiplication and the right action by multiplication composed with a braiding,

$$\rho_l = m \otimes \mathrm{id}_U \in \mathrm{Hom}(A \otimes A \otimes U, A \otimes U)$$

and

$$\rho_r^+ = (m \otimes \mathrm{id}_U) \circ (\mathrm{id}_A \otimes c_{U,A}) \quad \text{and} \quad \rho_r^- = (m \otimes \mathrm{id}_U) \circ \big(\mathrm{id}_A \otimes c_{A,U}^{-1}\big).$$

We call these functors *braided induction* functors. They have been introduced, under the name α-induction, in operator algebra theory [LR, X, BE]. For more details in a category-theoretic framework, we refer to [O, Sect. 5.1].

We pause to recall that [VZ] an *Azumaya algebra* A is an algebra for which the two functors $\alpha_A^{\pm}$ are equivalences of tensor categories. This should be compared to the textbook definition of an Azumaya algebra in the tensor category of modules over a commutative $\Bbbk$-algebra A, requiring in particular the morphism

$$\psi_A : A \otimes A^{\mathrm{opp}} \to \mathrm{End}(A)$$
$$a \otimes a' \mapsto (x \mapsto a \cdot x \cdot a')$$

to be an isomorphism of algebras. Indeed, in this situation for an Azumaya algebra A, we have the following chain of equivalences:

$$A\text{-bimod} \xrightarrow{\sim} A \otimes A^{\mathrm{opp}}\text{-mod} \xrightarrow{\psi_A} \mathrm{End}(A)\text{-mod} \xrightarrow{\text{Morita}} \mathcal{V}ect(\Bbbk).$$

We now introduce the properties of an algebra A to be symmetric and special.

Definition 3.1 Let $\mathcal{C}$ be a tensor category.

1. For $\mathcal{C}$ enriched over the category of $\Bbbk$-vector spaces, a *special algebra* in $\mathcal{C}$ is an object A of $\mathcal{C}$ that is endowed with an algebra structure (A, m, η) and a coalgebra structure (A, Δ, ε) such that

$$\varepsilon \circ \eta = \beta_1 \, \mathrm{id}_{\mathbf{1}} \quad \text{and} \quad m \circ \Delta = \beta_A \, \mathrm{id}_A$$

with invertible elements $\beta_1, \beta_A \in \Bbbk^{\times}$.

2. A *symmetric algebra* in $\mathcal{C}$ is an algebra (A, m, η) together with a morphism $\varepsilon \in \mathrm{Hom}(A, \mathbf{1})$ such that the two morphisms

$$\Phi_1 := \big[(\varepsilon \circ m) \otimes \mathrm{id}_{A^\vee}\big] \circ (\mathrm{id}_A \otimes b_A) \in \mathrm{Hom}(A, A^\vee) \quad \text{and} \tag{1}$$

$$\Phi_2 := \big[\mathrm{id}_{A^\vee} \otimes (\varepsilon \circ m)\big] \circ \big(\tilde{b}_A \otimes \mathrm{id}_A\big) \in \mathrm{Hom}\big(A, A^\vee\big) \tag{2}$$

are identical.

Special algebras are in particular separable, and as a consequence, their categories of modules and bimodules are semisimple. A class of examples of special Frobenius algebras is supplied by the Frobenius algebra structure on a Hopf algebra H in $\mathcal{C}$, provided that H is semisimple.

We now consider the case of a semisimple modular tensor category $\mathcal{C}$ and introduce, for any algebra A in $\mathcal{C}$, the square matrix $(Z_{ij})_{i,j\in I}$ with entries

$$Z_{ij}(A) := \dim_{\Bbbk} \mathrm{Hom}_{A|A}\big(\alpha_A^-(U_i), \alpha_A^+\big(U_j^\vee\big)\big),$$

where $\mathrm{Hom}_{A|A}$ stands for homomorphisms of bimodules. Identifying A-bimod with the tensor category of module endofunctors of A-mod, we see that the non-negative integers $Z_{ij}(A)$ only depend on the Morita class of A.

In this setting, and in case that the algebra A is symmetric and special, we can make the following statements.

Theorem 3.2 [FRS1, Theorem 5.1(i)] *For $\mathcal{C}$ a semisimple modular tensor category and A a special symmetric Frobenius algebra in $\mathcal{C}$, the morphism*

$$\sum_{i,j\in I} Z_{ij}(A)\chi_i \otimes \chi_j \in \mathrm{Hom}(\mathbf{1}, \mathcal{H}) \otimes_{\Bbbk} \mathrm{Hom}(\mathbf{1}, \mathcal{H}) \tag{3}$$

is invariant under the diagonal action of $\mathrm{SL}(2, \mathbb{Z})$.

Remarks 3.3

1. In conformal field theory, expression (3) has the interpretation of a partition function for bulk fields.
2. For semisimple tensor categories based on the sl(2) affine Lie algebra, an A–D–E pattern appears [KO].

We finally summarize a few other results that hold under the assumption that $\mathcal{C}$ is a semisimple modular tensor category and A a symmetric special Frobenius algebra in $\mathcal{C}$. To formulate them, we need the following ingredients: The *fusion algebra*

$$R_{\mathcal{C}} := K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \Bbbk$$

is a separable commutative algebra with a natural basis $\{[U_i]\}_{i\in I}$ given by the isomorphism classes of simple objects. The matrix S introduced in Definition 1.4 provides a natural bijection from the set of isomorphism classes of irreducible representations of $R_{\mathcal{C}}$ to I.

Theorem 3.4 [FRS1, Theorem 5.18] *For any special symmetric Frobenius algebra A, the vector space $K_0(A\text{-}mod) \otimes_{\mathbb{Z}} \Bbbk$ is an $R_{\mathcal{C}}$-module. The multiset* $\mathrm{Exp}(A\text{-}mod)$ *that contains the irreducible $R_{\mathcal{C}}$-representations, with their multiplicities in this $R_{\mathcal{C}}$-module, can be expressed in terms of the matrix $Z(A)$*:

$$\mathrm{Exp}(A\text{-}mod) = \mathrm{Exp}\big(Z(A)\big) := \big\{ i \in I \textit{ with multiplicity } Z_{ii}(A) \big\}.$$

The observation that the vector space $K_0(A\text{-mod}) \otimes_{\mathbb{Z}} \Bbbk$ has a natural basis provided by the classes of simple A-modules gives the following:

Corollary 3.5 *The number of isomorphism classes of simple A-modules equals* $\mathrm{tr}(Z(A))$.

The category A-bimod of A-bimodules has the structure of a tensor category. From the fact that A is a symmetric special Frobenius algebra, it follows [FS] that A-bimod inherits left and right dualities from $\mathcal{C}$. Hence the tensor product on A-bimod is exact, and thus $K_0(A\text{-bimod})$ is a ring. The corresponding $\Bbbk$-algebra can again be described in terms of the matrix $Z(A)$:

Theorem 3.6 [O, FRS2] *There is an isomorphism*

$$K_0(A\text{-}bimod) \otimes_{\mathbb{Z}} \Bbbk \cong \bigoplus_{i,j \in I} \mathrm{Mat}_{Z_{ij}(A)}(\Bbbk)$$

of $\Bbbk$-algebras, with $\mathrm{Mat}_n(\Bbbk)$ denoting the algebra of $\Bbbk$-valued $n \times n$-matrices.

Corollary 3.7 *The number of isomorphism classes of simple A-bimodules equals* $\mathrm{tr}(ZZ^{\mathrm{t}})$.

Theorem 3.8 [FFRS, Proposition 4.7] *Any A-bimodule is a subquotient of a bimodule of the form $\alpha_A^+(U) \otimes_A \alpha_A^-(V)$ for some pair of objects $U, V \in \mathcal{C}$.*

4 Outlook

We conclude this brief review with a few comments. First, all the results about algebra and representation theory in braided tensor categories that we have presented above are motivated by a construction of correlation functions of a rational conformal field theory as elements of vector spaces which are assigned by a topological field theory to a two-manifold. For details of this construction, we refer to [SFR] and the literature given there.

In the conformal field theory context the matrix Z describes the partition function of bulk fields. The three-dimensional topology involved in the RCFT construction provides in particular a motivation for using the different braidings which lead to the functors α_A^+ and α_A^- and to the definition of $Z(A)$.

To extend the results obtained in connection with rational conformal field theory to non-semisimple finite braided tensor categories remains a major challenge. Intriguing first results include, at the level of chiral data, a generalization of the Verlinde formula (see [GT] and references given there) and, at the level of partition functions, the bulk partition functions for logarithmic conformal field theories in the $(1, p)$-series found in [GR].

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Mutation Classes of 3 × 3 Generalized Cartan Matrices

Ahmet I. Seven

Abstract One of the recent developments in representation theory has been the introduction of cluster algebras by Fomin and Zelevinsky. It is now well known that these algebras are closely related with different areas of mathematics. A particular analogy exists between combinatorial aspects of cluster algebras and Kac–Moody algebras: roughly speaking, cluster algebras are associated with skew-symmetrizable matrices, while Kac–Moody algebras correspond to (symmetrizable) generalized Cartan matrices. In this paper, we describe an interplay between these two classes of matrices in size 3. In particular, we give a characterization of the mutation classes associated with the generalized Cartan matrices of size 3, generalizing results of Beineke–Bruestle–Hille.

Keywords Mutation · Skew-symmetrizable matrix · Generalized Cartan matrix

Mathematics Subject Classification (2010) Primary: 05E15, Secondary: 13F60 · 05C50 · 15B36 · 17B67

1 Introduction

One of the recent developments in representation theory has been the introduction of cluster algebras by Fomin and Zelevinsky to provide an algebraic framework for a study of Lusztig's canonical bases and positivity in algebraic groups. It is now well known that these algebras are also closely related with many different areas of mathematics (see, e.g., [6] for an account of these connections). A particular analogy exists between combinatorial aspects of cluster algebras and Kac–Moody algebras: roughly speaking, cluster algebras are associated with skew-symmetrizable matrices, while Kac–Moody algebras correspond to (symmetrizable) generalized Cartan matrices. The goal of this paper is to describe an interplay between these two classes of matrices in size 3, characterizing the skew-symmetrizable matrices associated with the generalized Cartan matrices. (We will deal with combinatorial aspects of these algebras but will not need or use their algebraic properties, including their definition.)

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To be more specific, we need some terminology. Let us recall that an integer matrix B is skew-symmetrizable if DB is skew-symmetric for some diagonal matrix D with positive diagonal entries. One of the main inventions in the theory of cluster algebras is an explicitly-defined, yet mysterious, operation, called mutation, on skew-symmetrizable matrices. More precisely, for any matrix index k, the mutation of a skew-symmetrizable matrix B in direction k is another skew-symmetrizable matrix $\mu_k(B) = B'$:

$$B' = \begin{cases} B'_{i,j} = -B_{i,j} & \text{if } i = k \text{ or } j = k \\ B'_{i,j} = B_{i,j} + \operatorname{sgn}(B_{i,k})[B_{i,k}B_{k,j}]_+ & \text{else} \end{cases}$$

(where we use the notation $[x]_+ = \max\{x, 0\}$ and $\operatorname{sgn}(x) = x/|x|$ with $\operatorname{sgn}(0) = 0$). Mutation is an involutive operation, so repeated mutations in all directions give rise to the *mutation-equivalence* relation on skew-symmetrizable matrices; each mutation-equivalence class uniquely determines, in particular, a cluster algebra. Therefore it is natural to ask for an explicit description and classification of these mutation classes.

Let us also recall a related combinatorial construction from [4]: for a skew-symmetrizable $n \times n$ matrix B, its *diagram* is defined to be the directed graph $\Gamma(B)$ whose vertices are the indices $1, 2, \ldots, n$ such that there is a directed edge from i to j if and only if $B_{ij} > 0$, and this edge is assigned the weight $|B_{ij}B_{ji}|$. Let us note that if B is not skew-symmetric, then the diagram $\Gamma(B)$ does not determine B as there could be several different skew-symmetrizable matrices whose diagrams are equal. The mutation μ_k can naturally be viewed as a transformation on diagrams (see [9, Sect. 2] for a description). In the particular case where B is skew-symmetric, the diagram $\Gamma(B)$ may be viewed as a quiver, and the corresponding mutation operation is also called quiver mutation [6].

On the other hand, Kac–Moody algebras correspond to (symmetrizable) generalized Cartan matrices. It has been observed, using structural properties of cluster algebras and related categorical methods, that some mutation-equivalence classes of skew-symmetrizable matrices are determined by generalized Cartan matrices [3, 4]. These are the mutation classes which contain a representative with an acyclic diagram (i.e., a diagram with no oriented cycles at all). To study such a correspondence in a more explicit and direct fashion, a notion of a *quasi-Cartan companion* has been introduced in [2]: a quasi-Cartan companion of a skew-symmetrizable matrix is a symmetrizable matrix whose diagonal entries are equal to 2 and whose off-diagonal entries differ only by signs. This notion has been successfully used in [2, 9] to describe the mutation classes associated with generalized Cartan matrices of finite and affine types. In this paper, we will deal with the next basic indefinite (wild) case to describe the mutation classes of skew-symmetrizable matrices associated with generalized Cartan matrices of size 3.

Our characterization (Theorem 2.6) of the mutation classes associated with generalized Cartan matrices of size 3 may be viewed as a generalization of the one given in [1] in the special case of skew-symmetric matrices. The description in [1] uses a third-degree polynomial, called the Markov constant (Definition 2.3). We give a natural interpretation of the Markov constant in terms of generalized Cartan matrices (Proposition 2.5). This allows us to work in the more general case of skew-symmetrizable matrices and suggests generalizations. The Markov constant appeared earlier in the literature, particularly on vector bundles, see, e.g., [8]. The relation between these contexts has not been particularly studied.

2 Mutation Classes of 3 × 3 (Skew-)Symmetrizable Matrices: Quasi-Cartan Companions and the Markov Constant

To give precise statements and proofs of our results we need to recall some more terminology; for details, we refer to [9, Sect. 2].

Let us first recall that a quasi-Cartan companion A of skew-symmetrizable matrix B is called *admissible* if it satisfies the following sign condition: for any cycle Z in $\Gamma(B)$, the product $\prod_{\{i,j\}\in Z}(-A_{i,j})$ over all edges of Z is negative if Z is oriented and positive if Z is non-oriented.

The main examples of admissible companions are the generalized Cartan matrices: if $\Gamma(B)$ is acyclic, i.e., has no oriented cycles at all, then the quasi-Cartan companion A with $A_{i,j} = -|B_{i,j}|$ for all $i \neq j$ is admissible. However, for an arbitrary skew-symmetrizable matrix B, an admissible quasi-Cartan companion may not exist; if it does exist, it is unique up to simultaneous sign changes in rows and columns.

Mutation operation on skew-symmetrizable matrices can be extended to their quasi-Cartan companions as follows:

Definition 2.1 Suppose that B is a skew-symmetrizable matrix and let A be a quasi-Cartan companion of B. Let k be a vertex in Γ. “The mutation of A at k” is the matrix A' defined as

$$A' = \begin{cases} A'_{k,k} = 2 & \\ A'_{i,k} = \operatorname{sgn}(B_{i,k})A_{i,k} & \text{if } i \neq k \\ A'_{k,j} = -\operatorname{sgn}(B_{k,j})A_{k,j} & \text{if } j \neq k \\ A'_{i,j} = A_{i,j} - \operatorname{sgn}(A_{i,k}A_{k,j})[B_{i,k}B_{k,j}]_+ & \text{if } i, j \neq k \end{cases}$$

Furthermore, $\det(A') = \mp\det(A)$. (Note that both matrices B and A are used.)

Let us note that A' may not be a quasi-Cartan companion of B; however, in the case A is admissible, A' is a quasi-Cartan companion. Furthermore, A' is admissible in some interesting cases, e.g. in the context of this paper (Proposition 2.8). More generally, we conjecture that admissibility is preserved in the mutation classes of acyclic diagrams. In this paper, we obtain first results to prove this conjecture in the indefinite case (for finite and affine types, it was obtained in [2, 9]). To state and prove these results, let us first give the following definition:

Definition 2.2 Suppose that Γ is the diagram of a 3 × 3 skew-symmetrizable matrix. Following [1], we say that Γ is cyclic if it is an oriented cycle (triangle). We

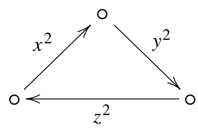

Fig. 1 Diagram of a 3×3 skew-symmetric matrix. (Note that quiver notation is used in [1])

say that Γ is mutation-cyclic if any diagram which is mutation-equivalent to Γ is cyclic; otherwise we call it mutation-acyclic.[1]

Mutation-acyclic diagrams represent the skew-symmetrizable matrices which correspond to generalized Cartan matrices [3]. This makes it natural to ask for an explicit characterization of these diagrams. For skew-symmetric matrices of size 3, a particular characterization has been obtained in [1] using a polynomial called the Markov constant. Let us recall the definition:

Definition 2.3 Suppose that B is a 3×3 skew-symmetric matrix whose diagram $\Gamma(B)$ is cyclic. Let x, y, z be the positive entries of B (so the weights of $\Gamma(B)$ are x^2, y^2 and z^2, see Fig. 1). We define the associated Markov constant as $C(B) = C(x, y, z) = x^2 + y^2 + z^2 - xyz$.

Note that $C(B)$ is invariant under simultaneous permutations of rows and columns (i.e. it is invariant under permutations of the vertices in $\Gamma(B)$).

Let us also recall from [1] how skew-symmetric matrices with mutation-acyclic diagrams are characterized by the Markov constant:

Theorem 2.4 [1, Theorem 1.1] *Suppose that B is a skew-symmetric (integer) matrix as in Definition 2.3 such that $\Gamma(B)$ is cyclic. Then the following are equivalent:*

(1) *$\Gamma(B)$ is mutation-acyclic.*
(2) *The Markov constant satisfies $C(x, y, z) > 4$ or $\min\{x, y, z\} < 2$.*
(3) *The Markov constant satisfies $C(x, y, z) > 4$, or the triple (x, y, z) is in the following list (where we assume that $x \geq y \geq z$):*
    (a) $C(x, y, z) = 0 : (x, y, z) = (0, 0, 0)$,
    (b) $C(x, y, z) = 1 : (x, y, z) = (1, 0, 0)$,
    (c) $C(x, y, z) = 2 : (x, y, z) = (1, 1, 0)$ *or* $(1, 1, 1)$,
    (d) $C(x, y, z) = 4 : (x, y, z) = (2, 0, 0)$ *or* $(2, 1, 1)$.

We would like to generalize this theorem to skew-symmetrizable matrices. However such a generalization is not immediate because the Markov constant is not defined for non-skew-symmetric matrices; it is also not defined for skew-symmetric matrices whose diagrams are acyclic. We will achieve this generalization by interpreting the Markov constant in terms of quasi-Cartan companions as follows:

[1] The term *cluster-acyclic* is used in [1].

Proposition 2.5 *Suppose that B is a skew-symmetric matrix of size* 3 *such that $\Gamma(B)$ is cyclic. Let A be an admissible quasi-Cartan companion of B. Then* $\det A = 2(4 - C(B))$ (*where $C(B)$ is the associated Markov constant as defined above*).

This statement follows from a direct computation.

We then characterize 3×3 skew-symmetrizable matrices with mutation-acyclic diagrams using determinants of associated quasi-Cartan companions:

Theorem 2.6 *Suppose that B is a skew-symmetrizable matrix of size* 3 *and let A be an admissible quasi-Cartan companion of B. Then $\Gamma(B)$ is mutation-acyclic if and only if one of the following holds*:

(i) $\det(A) > 0$, *and A is positive*,
(ii) $\det(A) = 0$, *and A is semipositive of corank* 1,
(iii) $\det(A) < 0$.

Let us note that parts (i) and (ii) occur if and only if $\Gamma(B)$ is mutation-equivalent to a Dynkin and an extended Dynkin diagram respectively [2, 9] (here a Dynkin diagram is an orientation of a Dynkin graph, see [9, Sect. 2]). For the convenience of the reader, let us give the special case of the statement for the indefinite type:

Corollary 2.7 *Suppose that B is a skew-symmetrizable matrix of size* 3 *whose diagram is not mutation-equivalent to a Dynkin or extended Dynkin diagram. Then $\Gamma(B)$ is mutation-acyclic if and only if* $\det(A) < 0$.

Let us remark that determinant of a quasi-Cartan companion, or Markov constant, does not determine the mutation class of a skew-symmetrizable matrix.

To prove the theorem, we need some preliminary statements. First we show the invariance of admissibility property:

Proposition 2.8 *Suppose that B is a skew-symmetrizable matrix of size* 3 *and let A be an admissible quasi-Cartan companion of B. Let A' be the mutation of A at k. Then A' is also admissible.*

Proof To prove, let us note that A' is a quasi-Cartan companion of $\mu_k(B) = B'$. If $\Gamma(B') = \Gamma'$ is a tree, then A' is admissible because admissibility is defined by conditions on cycles. Thus we can assume that Γ' is a cycle (triangle). Let us denote by A'' the quasi-Cartan companion of $\mu_k(B') = B$ obtained by mutating A' at k. Then, it follows from the definition of the mutation operation that A'' is equal to A up to a simultaneous sign change at kth row and column. In particular, A'' is an admissible quasi-Cartan companion of B. We will obtain a contradiction to this if A' is not admissible. For this purpose, let us assume to the contrary that A' is not an admissible quasi-Cartan companion of Γ'. Let us first consider the case where Γ' is oriented. Then, applying sign changes if necessary, we can assume that the signs of the entries outside the diagonal of A' are negative. More explicitly, let the vertices of Γ' be i, j, k, and let $\text{sgn}(A_{i,j}) = \text{sgn}(A_{j,k}) = \text{sgn}(A_{i,k}) = -1$. Then, it follows from the definitions of mutations that $|B_{i,j}| \neq |A''_{i,j}|$, implying that A'' is not a quasi-Cartan companion of B, a contradiction. The case where Γ' is non-oriented is considered similarly. This completes our proof. □

Proposition 2.9 *Suppose that A is a 3×3 generalized Cartan matrix of indefinite type. Then* $\det(A) < 0$.

This statement follows from direct computation. Let us give a proof here for

$$A = \begin{pmatrix} 2 & -a & -c \\ -a' & 2 & -b \\ -c' & -b' & 2 \end{pmatrix}$$

where $a, b, c, a', b', c' > 0$, i.e. where the Dynkin graph of A is a cycle (the remaining case where the Dynkin graph of A is a tree is done similarly). Since A is of indefinite type, we can assume without loss of generality that $bb' \geq 2$ (otherwise A is of affine type $A_1^{(1)}$). Expanding the determinant of A along the first row, we have

$$\det(A) = 2(4 - bb') - a(2a' + bc') - c(2c' + a'b').$$

Since the entries of A are integers, we have $2a' + bc', 2c' + a'b' \geq 3$, also $4 - bb' \leq 2$. Then $\det(A) < 0$.

Let us now give the following technical statement from [7], which provides (skew)-symmetrization by conjugation. It will be very useful to us.

Lemma 2.10 *Let B be a skew-symmetrizable (integer) matrix. Then there exists a diagonal matrix H with positive diagonal entries such that HBH^{-1} is skew-symmetric. Furthermore, the matrix $S(B) = (S_{ij}) = HBH^{-1}$ is uniquely determined by B. Specifically, the matrix entries of $S(B)$ are given by*

$$S_{ij} = \mathrm{sgn}(B_{ij})\sqrt{|B_{ij}B_{ji}|}. \tag{2.1}$$

Furthermore, for any matrix index k, we have $S(\mu_k(B)) = \mu_k(S(B))$.

The statement also holds for symmetrizable matrices, replacing B by a quasi-Cartan companion A respectively. The matrix H can be taken as $D^{1/2}$ where D is a skew-symmetrizing (resp. symmetrizing) matrix for B (resp. A).

We will also need the following technical result:

Lemma 2.11 [1, Lemma 3.3(b)] *Suppose that B is a 3×3 skew-symmetric matrix with real entries. If $\Gamma(B)$ is mutation-cyclic, then either $C(B) < 4$ or $C(B) = 4$ and $\Gamma(B)$ is mutation-equivalent to a diagram whose weights are $u, u, 4$ for some $u \geq 4$.*

We are now ready to prove Theorem 2.6. Let us first assume that $\Gamma(B)$ is mutation-equivalent to an acyclic diagram. Then, by Proposition 2.8, it has an admissible companion A which is obtained by a sequence of mutations from a generalized Cartan matrix say A'. In particular, A and A' are equivalent [9, Definition 2.13], so $\det(A) = \det(A')$. If A' is of finite or affine type, then part (i) or (ii) holds respectively by [2, 9]; if A' is indefinite type, then $\det(A') = \det(A) < 0$ by Proposition 2.9. For the converse, let us assume that $\Gamma = \Gamma(B)$ is mutation-cyclic, and let A be an admissible quasi-Cartan companion of B. Then, by [9, Theorem 3.1], neither (i) nor (ii) holds (otherwise Γ is mutation-equivalent to a Dynkin or extended Dynkin diagram). To complete the proof of the theorem, we will show that $\det(A) \geq 0$ and if $\det(A) = 0$, then A is not semipositive of corank 1. This can be obtained from Lemma 2.11 as follows: first note that, by Lemma 2.10, the skew-symmetric matrix $S(B)$ is also mutation-cyclic. Then, by Lemma 2.11, we have that $C(S(B)) \leq 4$ (so $\det(A) \geq 0$) and if $C(S(B)) = 4$ (equivalently $\det(A) = 0$), then $S(B)$ is mutation-equivalent to a skew-symmetric matrix $S(B')$ whose diagram has weights $4, u, u$ with $u \geq 4$. Then it follows from a direct check that any admissible quasi-Cartan companion of B' (in particular the ones that can be obtained from A by a sequence of mutations) is not semipositive. This completes the proof of the theorem.

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Contractions and Polynomial Lie Algebras

Benjamin J. Wilson

Abstract Let $\mathfrak{g}$ denote a Lie algebra over $\mathbb{k}$, and let B denote a commutative unital $\mathbb{k}$-algebra. The tensor product $\mathfrak{g} \otimes_{\mathbb{k}} B$ carries the structure of a Lie algebra over $\mathbb{k}$ with Lie bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab, \quad x, y \in \mathfrak{g}, \ a, b \in B.$$

If C_0 denotes the quotient of the polynomial algebra $\mathbb{k}[t]$ by the ideal generated by some power of t, then $\mathfrak{g} \otimes C_0$ is called a polynomial Lie algebra.

In this contribution, $\mathfrak{g} \otimes C_0$ is shown to be a contraction of $\mathfrak{g} \otimes C$, where C is a semisimple commutative unital algebra. The contraction is exploited to derive a reducibility criterion for the universal highest-weight modules of $\mathfrak{g} \otimes C_0$, via contraction of the Shapovalov form. This yields an alternative derivation of the reducibility criterion, obtained by the author in previous work.

Keywords Highest-weight theory · Contraction · Deformation · Polynomial Lie algebra

Mathematics Subject Classification (2010) 17B10 · 17B99

1 Introduction

Let $\mathbb{k}$ denote a field of characteristic 0. Let $\mathfrak{g}$, $\mathfrak{g}_0$ denote Lie algebra structures on a $\mathbb{k}$-vector space V with Lie brackets $[\cdot, \cdot]$ and $[\cdot, \cdot]_0$, respectively. A *contraction* of $\mathfrak{g}$ to $\mathfrak{g}_0$ is an algebraic map

$$\Psi : \mathbb{k}^\times \to \operatorname{End}_{\mathbb{k}} V, \quad z \mapsto \Psi_z,$$

such that

(i) $\det \Psi_z \neq 0$ for all $z \in \mathbb{k}^\times$;

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(ii) $\Psi_z{}^{-1}[\Psi_z u, \Psi_z v]$ is polynomial in z for any $u, v \in V$ and $z \in \mathbb{k}^\times$, and

$$\left(\Psi_z{}^{-1}[\Psi_z u, \Psi_z v]\right)_{z=0} = [u, v]_0.$$

If such a Ψ exists, then $\mathfrak{g}_0$ is said to be a contraction of $\mathfrak{g}$. Let $\mathfrak{g}$ denote a Lie algebra over $\mathbb{k}$, and let B denote a commutative unital $\mathbb{k}$-algebra. The tensor product $\mathfrak{g} \otimes_{\mathbb{k}} B$ carries the structure of a Lie algebra over $\mathbb{k}$ with Lie bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab, \quad x, y \in \mathfrak{g},\ a, b \in B. \tag{1}$$

A triangular decomposition (cf. Sect. 3, here $\mathfrak{h}_0 = \mathfrak{h}$) of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$, naturally defines a triangular decomposition of $\mathfrak{g} \otimes B$,

$$\mathfrak{g} \otimes B = (\mathfrak{g}_- \otimes B) \oplus (\mathfrak{h} \otimes B) \oplus (\mathfrak{g}_+ \otimes B),$$

where $\mathfrak{h} \cong \mathfrak{h} \otimes \mathbb{k}1_B \subset \mathfrak{h} \otimes B$ acts diagonally on $\mathfrak{g} \otimes B$ under the adjoint action. Fix an integer $l > 1$, and let C_0 denote the quotient of the polynomial algebra $\mathbb{k}[t]$ by the ideal generated by t^l. The Lie algebra $\mathfrak{g} \otimes C_0$ is called a *polynomial Lie algebra*. Write C for the semisimple algebra formed by l-copies of $\mathbb{k}$. In Sect. 2, we construct a contraction of the Lie algebra $\mathfrak{g} \otimes C$ to $\mathfrak{g} \otimes C_0$, using a degeneration θ_z of the semisimple algebra C to the truncated polynomial ring C_0. In this paper, a reducibility criterion for the universal highest-weight modules of $\mathfrak{g} \otimes C_0$ (i.e. for the Verma modules) is derived, using the contraction of Lie algebras, under mild hypotheses on $\mathfrak{g}$.

Write $\mathfrak{g}_\pm = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}^{\pm\alpha}$ for the decomposition of $\mathfrak{g}_+, \mathfrak{g}_-$ into eigenspaces for the adjoint action of the diagonal subalgebra. Assume that $\mathfrak{g}$ is *non-degenerately paired*, i.e. that for all $\alpha \in \Delta_+$, a non-degenerate bilinear form

$$(\cdot|\cdot)_\alpha : \mathfrak{g}^\alpha \times \mathfrak{g}^{-\alpha} \to \mathbb{k}$$

and a non-zero $\mathbf{h}(\alpha) \in \mathfrak{h}$ are given such that

$$[x, y] = (x|y)_\alpha \mathbf{h}(\alpha), \quad x \in \mathfrak{g}^\alpha,\ y \in \mathfrak{g}^{-\alpha}. \tag{2}$$

The symmetrizable Kac–Moody Lie algebras, the Virasoro algebra and the Heisenberg algebra are examples of Lie algebras with non-degenerate pairings. The principal result of this contribution is the following theorem.

Theorem 3 *For any* $\Lambda \in (\mathfrak{h} \otimes C_0)^*$, *the Verma module* $\mathrm{M}(\Lambda)$ *for* $\mathfrak{g} \otimes C_0$ *is reducible if and only if*

$$\left\langle \Lambda, \mathbf{h}(\alpha) \otimes t^{l-1} \right\rangle = 0$$

for some $\alpha \in \Delta_+$.

Theorem 3 is a corollary of Theorem 32, which gives a formula for the determinant of the restrictions of the Shapovalov form to an arbitrary weight space. This determines which weight spaces of the Verma module contribute to the maximal submodule. The same result was obtained by the author, using more pedestrian methods, in previous work [6]. The present work showcases a more broadly applicable approach and is the prototype for the author's investigations of applications of contractions in representation theory, to appear in later work.

The paper is structured as follows. Section 2 constructs a contraction of $\mathfrak{g} \otimes C$ to $\mathfrak{g} \otimes C_0$ from a certain degeneration of C to C_0. Section 3 provides the necessary background on triangular decompositions, non-degenerate pairings of Lie algebras and highest-weight theory. Finally, Sect. 4 relates the Shapovalov determinants of $\mathfrak{g} \otimes C$ and $\mathfrak{g} \otimes C_0$ using the contraction, and derives sufficient information about the former to obtain a closed formula for the latter.

The reader is referred to [4] for an extensive review of contractions of Lie groups and Lie algebras. In [2], another contraction of a semisimple Lie algebra to the polynomial Lie algebra is described. Polynomial Lie algebras are also known as *truncated current Lie algebras* and as *Takiff algebas*. The work [6] reviews previous studies of polynomial Lie algebras and their applications.

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2 The Polynomial Lie Algebra as a Contraction

The semisimple algebra C is isomorphic to any quotient of the polynomial algebra $\mathbb{k}[t]$ by a degree-l polynomial with pair-wise distinct roots in $\mathbb{k}$. In this section, a degeneration θ_z of C to C_0 is constructed which formalizes the intuition that C_0 is "the limit" of C as the distinct roots of the polynomial defining the quotient approach a common point, 0. Using θ_z, a contraction $\Psi_z = \mathrm{id}_{\mathfrak{g}} \otimes \theta_z$ of $\mathfrak{g} \otimes C$ to $\mathfrak{g} \otimes C_0$ may be constructed for any Lie algebra $\mathfrak{g}$.

Let $\mathbf{V}_l$ denote an l-dimensional $\mathbb{k}$-vector space with basis $\{v_i | 1 \leq i \leq l\}$. Write C for the algebra structure on $\mathbf{V}_l$ for which the v_i are orthogonal idempotents, and write $\cdot$ for its multiplication. Fix an l-tuple $(\zeta_i) \in \mathbb{k}^l$ of pair-wise distinct scalars, and for any $z \in \mathbb{k}$, let C_z denote the quotient of the algebra $\mathbb{k}[t]$ by $\prod_{i=1}^{l}(t - z\zeta_i)$. Identify the vector spaces C_z and $\mathbf{V}_l$ via $t^{i-1} \leftrightarrow v_i$ for $1 \leq i \leq l$, and write $\underset{z}{\cdot}$ for the multiplication of C_z. For $1 \leq i \leq l$, let $\rho_i : \mathbb{k} \to \mathbf{V}_l$ denote the inclusion of $\mathbb{k}$-algebras defined by $\rho_i : 1 \mapsto v_i$. By abuse of notation, denote by ρ_i also the maps it induces, e.g. for a Lie algebra $\mathfrak{g}$,

$$\rho_i : \mathfrak{g} \to \mathfrak{g} \otimes C, \qquad \rho_i : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g} \otimes C), \qquad \rho_i : \mathrm{S}(\mathfrak{h}) \to \mathrm{S}(\mathfrak{h} \otimes C),$$

where $\mathrm{U}(\cdot)$ denotes the enveloping algebra of a Lie algebra, and $\mathrm{S}(\cdot)$ denotes the symmetric algebra associated to a vector space. For $z \in \mathbb{k}^\times$ and $1 \leq i \leq l$, let

$$T_i(z) = \prod_{j \neq i} \frac{(t - z\zeta_j)}{(z\zeta_i - z\zeta_j)}. \tag{4}$$

Let $\sigma_i = \prod_{j \neq i}(\zeta_i - \zeta_j)$ and write $\epsilon_k(\hat{\zeta}_i)$ for the degree-k elementary symmetric function in the $(l-1)$-variables ζ_j, $j \neq i$. Then

$$T_i(z) = {\sigma_i}^{-1} \sum_{k=1}^{l} (-1)^{l-k} z^{-(k-1)} \epsilon_{l-k}(\hat{\zeta}_i) v_k \tag{5}$$

modulo the identification of the vector spaces C_z and $\mathbf{V}_l$. Notice that $T_i(z)$ is a polynomial in z^{-1} of degree $(l-1)$.

Lemma 6 *For $z \in \Bbbk^\times$, $\{T_i(z) | 1 \leq i \leq l\}$ is a set of orthogonal idempotents in C_z.*

Proof If $i \neq j$, then $T_i(z) \underset{z}{\cdot} T_j(z) = 0$ since $(t - z\zeta_j) \underset{z}{\cdot} T_j(z) = 0$, while the same relation, recast as $t \underset{z}{\cdot} T_j(z) = z\zeta_j \underset{z}{\cdot} T_j(z)$, shows that $T_j(z)$ is idempotent under the multiplication $\underset{z}{\cdot}$ of C_z. □

For $z \in \Bbbk$, define $\theta_z \in \operatorname{End}_\Bbbk \mathbf{V}_l$ where

$$(\theta_z)_{i,j} = (z\zeta_i)^{j-1}, \quad 1 \leq i, j \leq l,$$

as a matrix expressed with respect to the basis $\{v_i | 1 \leq i \leq l\}$.

Proposition 7

(i) $\det \theta_z = z^{\binom{l}{2}}$ *up to a non-zero scalar multiple*;
(ii) *for $z \in \Bbbk^\times$, $\theta_z : C_z \to C$ is an isomorphism of algebras such that*

$$\theta_z : T_i(z) \mapsto v_i, \quad 1 \leq i \leq l;$$

(iii) *for any $u, v \in \boldsymbol{V}_l$ and $z \in \Bbbk^\times$, ${\theta_z}^{-1}(\theta_z u \cdot \theta_z v)$ is polynomial in z, and*

$$u \cdot_0 v = \left[{\theta_z}^{-1}(\theta_z u \cdot \theta_z v)\right]_{z=0}.$$

Proof The first part follows from the Vandermonde determinant formula. Let $z \in \Bbbk^\times$. For the second, it is sufficient to show that $\theta_z : T_i(z) \mapsto v_i$ for all $1 \leq i \leq l$, since this defines an isomorphism of algebras by Lemma 6. We have

$$\begin{aligned} \theta_z T_i(z) &= {\sigma_i}^{-1} \sum_{k=1}^{l} (-1)^{l-k} z^{-(k-1)} \epsilon_{l-k}(\hat{\zeta}_i) \sum_{j=1}^{l} (z\zeta_j)^{k-1} v_j \\ &= {\sigma_i}^{-1} \sum_{j=1}^{l} \left(\sum_{k=1}^{l} (-1)^{l-k} \epsilon_{l-k}(\hat{\zeta}_i) {\zeta_j}^{k-1} \right) v_j \end{aligned}$$

where, for each $1 \leq j \leq l$, the bracketed quantity is the expansion of $[\prod_{k \neq i}(q - \zeta_k)]_{q=\zeta_j}$, which equals σ_i if $i = j$ and 0 otherwise. Hence $\theta_z T_i(z) = v_i$. The third part is immediate from the second. □

Corollary 8 *For any $\Bbbk$-Lie algebra $\mathfrak{g}$, $\Psi_z = \mathrm{id}_{\mathfrak{g}} \otimes \theta_z$ is a contraction of $\mathfrak{g} \otimes C$ to $\mathfrak{g} \otimes C_0$ as $\Bbbk$-Lie algebras.*

For any $u, v \in \mathbf{V}_l$, let $m_i(u, v) \in \mathbf{V}_l$, $i \in \mathbb{Z}$, be defined by

$$u \underset{z}{\cdot} v = \sum_{i \geq 0} m_i(u, v) z^i.$$

Let $\mathbf{C}_z$ denote the unital $\Bbbk[z, z^{-1}]$-algebra with underlying space $\mathbf{V}_l \otimes \Bbbk[z, z^{-1}]$ and multiplication, which by abuse of notation is denoted $\underset{z}{\cdot}$, defined by $\Bbbk$-linear extension of

$$u \otimes z^j \underset{z}{\cdot} v \otimes z^k = \sum_{i \geq 0} m_i(u, v) z^{i+j+k}$$

for all $u, v \in \mathbf{V}_l$ and $j, k \in \mathbb{Z}$. Thus $\mathbf{C}_z$ is a "generic version" of the $\Bbbk$-algebras C_z, $z \in \Bbbk^\times$. Let $C \otimes \Bbbk[z, z^{-1}]$ denote the unital $\Bbbk[z, z^{-1}]$-algebra with the same underlying space and multiplication $\cdot$ defined by the multiplication of C and extension of scalars. Let

$$\iota : \mathbf{V}_l \to \mathbf{V}_l \otimes \Bbbk\left[z, z^{-1}\right], \quad v \mapsto v \otimes 1, \ v \in \mathbf{V}_l,$$

denote the inclusion of $\Bbbk$-vector spaces. Then ι is an injection of unital $\Bbbk$-algebras $C \to C \otimes \Bbbk[z, z^{-1}]$, and

$$u \cdot_0 v = \left[\iota(u) \underset{z}{\cdot} \iota(v)\right]_{z=0}, \quad u, v \in \mathbf{V}_l. \tag{9}$$

Finally, by Proposition 7, the linear map $\mathbf{C}_z \to C \otimes \Bbbk[z, z^{-1}]$ defined by

$$v_j \otimes z^k \mapsto \sum_{i=1}^{l} v_i \otimes z^{j+k-1}$$

for all $1 \leq i \leq l$ and $k \in \mathbb{Z}$, is an isomorphism of unital $\Bbbk[z, z^{-1}]$-algebras; by abuse of notation, denote this isomorphism by θ_z.

3 Triangular Decompositions and Highest-Weight Theory

The definition of triangular decomposition used here is a modification of the definition of Moody and Pianzola [3]. This section, which paraphrases that text, indicates the consequent modifications of the notions of weight-modules, highest-weight modules, Verma modules and the Shapovalov form.

3.1 Triangular Decompositions

Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. A *triangular decomposition* of $\mathfrak{g}$ is specified by a pair of non-zero abelian subalgebras $\mathfrak{h}_0 \subset \mathfrak{h}$, a pair of distinguished non-zero subalgebras $\mathfrak{g}_+, \mathfrak{g}_-$ and an anti-involution $\omega : \mathfrak{g} \to \mathfrak{g}$, such that:

(i) $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$;
(ii) the subalgebra $\mathfrak{g}_+$ is a non-zero weight module for $\mathfrak{h}_0$ under the adjoint action,

$$\mathfrak{g}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}^\alpha,$$

with weights $\Delta_+ \subset \mathfrak{h}_0^*$ all non-zero;
(iii) $\omega|_\mathfrak{h} = \mathrm{id}_\mathfrak{h}$ and $\omega(\mathfrak{g}_+) = \mathfrak{g}_-$;
(iv) the semigroup with identity $\mathcal{Q}_+$, generated by Δ_+ under addition, is freely generated by a finite subset $\{\alpha_j\}_{j \in \mathrm{J}} \subset \mathcal{Q}_+$ consisting of linearly independent elements of $\mathfrak{h}_0^*$.

The anti-involution ensures a decomposition of $\mathfrak{g}_- = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}^{-\alpha}$, and $\mathfrak{g}^{-\alpha} = \omega(\mathfrak{g}^\alpha)$ for all $\alpha \in \Delta_+$. Consider $\mathcal{Q}_+$ to be partially ordered in the usual manner, i.e. for $\gamma, \gamma' \in \mathcal{Q}_+$,

$$\gamma \leq_{\mathcal{Q}_+} \gamma' \quad \Longleftrightarrow \quad (\gamma' - \gamma) \in \mathcal{Q}_+.$$

We assume that all root spaces are finite-dimensional. For clarity, a Lie algebra with triangular decomposition may be referred to as a five-tuple $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$. In [3], the set J is not required to be finite, root spaces may be infinite-dimensional, and $\mathfrak{h}_0 = \mathfrak{h}$. We distinguish between $\mathfrak{h}_0$ and $\mathfrak{h}$ in order to include Example 10. As described in [3], the Kac–Moody Lie algebras, the Virasoro algebra and the Heisenberg algebra are examples of Lie algebras with triangular decompositions.

Example 10 Let $\mathfrak{g}$ be a $\Bbbk$-Lie algebra with triangular decomposition, denoted as above, and let B be a commutative unital $\Bbbk$-algebra. Write $\hat{\mathfrak{g}} = \mathfrak{g} \otimes_\Bbbk B$, and similarly for the subalgebras of $\mathfrak{g}$. Then $\hat{\mathfrak{g}}$ is a $\Bbbk$-Lie algebra with Lie bracket (1) and contains $\mathfrak{g}$ as a subalgebra via $x \mapsto x \otimes 1$. Moreover, $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{g}}_+$ and $\mathfrak{h}_0 \cong \mathfrak{h}_0 \otimes \Bbbk 1_B \subset \hat{\mathfrak{h}}$ are non-zero abelian subalgebras of $\hat{\mathfrak{g}}$. The subalgebra $\hat{\mathfrak{g}}_+$ is a weight module for $\mathfrak{h}_0$ with weights coincident with the weights Δ_+ of the $\mathfrak{h}_0$-module $\mathfrak{g}_+$, and $(\hat{\mathfrak{g}}_+)^\alpha = \widehat{(\mathfrak{g}_+^\alpha)}$. Thus, $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ have the same positive roots Δ_+ and weight lattice $\mathcal{Q}_+$. The anti-involution ω of $\hat{\mathfrak{g}}$ is given by B-linear extension

$$\omega : x \otimes b \mapsto \omega(x) \otimes b, \quad x \in \mathfrak{g},\ b \in B,$$

and fixes $\hat{\mathfrak{h}}$ point-wise. Thus $(\hat{\mathfrak{g}}, \mathfrak{h}_0, \hat{\mathfrak{h}}, \hat{\mathfrak{g}}_+, \omega)$ is a $\Bbbk$-Lie algebra with triangular decomposition. Note that, in general, this triangular decomposition is not non-degenerately paired over the field $\Bbbk$.

3.2 Highest-Weight Modules

For the remainder of the section, $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$ denotes a Lie algebra with triangular decomposition. A $\mathfrak{g}$-module N is *weight* if the action of $\mathfrak{h}_0$ on N is diagonalizable, i.e.

$$N = \bigoplus_{\chi\in\mathfrak{h}_0^*} N^\chi, \quad h|_{N^\chi} = \chi(h) \text{ for all } h \in \mathfrak{h}_0, \chi \in \mathfrak{h}_0^*.$$

A non-zero vector $v \in N$ is a *highest-weight vector* if

(i) $\mathfrak{g}_+ \cdot v = 0$;
(ii) there exists $\Lambda \in \mathfrak{h}^*$ such that $h \cdot v = \Lambda(h)v$ for all $h \in \mathfrak{h}$.

The unique functional $\Lambda \in \mathfrak{h}^*$ is called the *highest weight* of the highest-weight vector v. A weight $\mathfrak{g}$-module N is called *highest weight* (of highest weight Λ) if there exists a highest-weight vector $v \in N$ (of highest weight Λ) that generates it. Let $\Lambda \in \mathfrak{h}^*$, and consider the one-dimensional vector space $\Bbbk \mathrm{v}_\Lambda$ as an $(\mathfrak{h} \oplus \mathfrak{g}_+)$-module via

$$\mathfrak{g}_+ \cdot \mathrm{v}_\Lambda = 0; \qquad h \cdot \mathrm{v}_\Lambda = \Lambda(h)\mathrm{v}_\Lambda, \quad h \in \mathfrak{h}.$$

The induced module

$$\mathrm{M}(\Lambda) = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{h}\oplus\mathfrak{g}_+)} \Bbbk \mathrm{v}_\Lambda$$

is called the *Verma module* of highest weight Λ.

Proposition 11 [3] *For any* $\Lambda \in \mathfrak{h}^*$,

(i) *up to scalar multiplication, there is a unique epimorphism from* $\mathrm{M}(\Lambda)$ *to any highest-weight module of highest-weight* Λ, *i.e.* $\mathrm{M}(\Lambda)$ *is the universal highest-weight module of highest-weight* Λ;
(ii) $\mathrm{M}(\Lambda)$ *is a free rank one* $\mathrm{U}(\mathfrak{g}_-)$*-module.*

3.3 The Shapovalov Form

It follows from the Poincaré–Birkhoff–Witt (PBW) theorem that $\mathrm{U}(\mathfrak{g})$ may be decomposed

$$\mathrm{U}(\mathfrak{g}) = \mathrm{U}(\mathfrak{h}) \oplus \{\mathfrak{g}_-\mathrm{U}(\mathfrak{g}) + \mathrm{U}(\mathfrak{g})\mathfrak{g}_+\}$$

as a direct sum of vector spaces. Let $\mathbf{q} : \mathrm{U}(\mathfrak{g}) \to \mathrm{S}(\mathfrak{h})$ denote the projection onto the first summand parallel to the second. Define

$$\mathbf{F} = \mathbf{F}_\mathfrak{g} : \mathrm{U}(\mathfrak{g}_-) \times \mathrm{U}(\mathfrak{g}_-) \to \mathrm{S}(\mathfrak{h}) \quad \text{via } \mathbf{F}(y_1, y_2) = \mathbf{q}\big(\omega(y_1)y_2\big), \ y_1, y_2 \in \mathrm{U}(\mathfrak{g}_-).$$

The bilinear form $\mathbf{F}$ is called the *Shapovalov form*. Distinct $\mathfrak{h}_0$-weight spaces of $U(\mathfrak{g}_-)$ are orthogonal with respect to $\mathbf{F}$, and so the study of $\mathbf{F}$ on $U(\mathfrak{g}_-)$ reduces to the study of the restrictions

$$\mathbf{F}^\chi = \mathbf{F}^\chi_\mathfrak{g} : U(\mathfrak{g}_-)^{-\chi} \times U(\mathfrak{g}_-)^{-\chi} \to S(\mathfrak{h}), \quad \chi \in \mathcal{Q}_+.$$

Any $\Lambda \in \mathfrak{h}^*$ extends uniquely to a map $S(\mathfrak{h}) \to \mathbb{k}$; write $\mathbf{F}^\chi(\Lambda)$ for the composition of $\mathbf{F}^\chi$ with this extension, and write $\operatorname{Rad}\mathbf{F}^\chi(\Lambda)$ for its radical. The importance of the Shapovalov form stems from the following fact.

Proposition 12 *Let* $\chi \in \mathcal{Q}_+$, $\Lambda \in \mathfrak{h}^*$. *Then* $\operatorname{Rad}\boldsymbol{F}^\chi(\Lambda) \subset M(\Lambda)^{\Lambda|_{\mathfrak{h}_0}-\chi}$ *is the* $\Lambda|_{\mathfrak{h}_0} - \chi$ *weight space of the maximal submodule of the Verma module* $M(\Lambda)$.

3.4 Partitions and the Poincaré–Birkhoff–Witt Monomials

Let $\mathcal{B}$ be a set parameterizing a root-basis (i.e. an $\mathfrak{h}_0$-weight basis) of $\mathfrak{g}_-$, via

$$\mathcal{B} \ni \gamma \quad \leftrightarrow \quad y(\gamma) \in \mathfrak{g}_-.$$

Define a function $r : \mathcal{B} \to \Delta_+$ by declaring $y(\gamma) \in \mathfrak{g}^{-r(\gamma)}$ for all $\gamma \in \mathcal{B}$. A *partition* is a finite multiset with elements from $\mathcal{B}$; write $\mathcal{P} = \mathcal{P}_\mathfrak{g}$ for the set of all partitions. Set notation is used for multisets throughout. In particular, the *length* $|\lambda|$ of a partition $\lambda \in \mathcal{P}$ is the number of elements of λ, counting all repetitions. Fix some ordering of the basis $\{y(\gamma)|\gamma \in \mathcal{B}\}$ of $\mathfrak{g}_-$; for any $\lambda \in \mathcal{P}$, let

$$y(\lambda) = y(\lambda_1) \cdots y(\lambda_k) \in U(\mathfrak{g}_-) \tag{13}$$

where $k = |\lambda|$, and $(\lambda_i)_{1 \le i \le k}$ is an enumeration of the entries of λ such that (13) is a PBW monomial with respect to the basis ordering. By the PBW Theorem, $U(\mathfrak{g}_-)$ has a basis $\{y(\lambda)|\lambda \in \mathcal{P}\}$. For any $\lambda \in \mathcal{P}$ and $\alpha \in \Delta_+$, write

$$r(\lambda) = \sum_{\gamma \in \lambda} r(\gamma); \quad \lambda^\alpha = \left\{\gamma \in \lambda | r(\gamma) = \alpha\right\}.$$

For any $\chi \in \mathcal{Q}_+$, let $\mathcal{P}^\chi_\mathfrak{g} = \{\lambda \in \mathcal{P}_\mathfrak{g} | r(\lambda) = \chi\}$, and for $\mathcal{S} \subset \mathcal{P}$, write

$$\operatorname{span}\mathcal{S} = \operatorname{span}\left\{y(\lambda)|\lambda \in \mathcal{S}\right\}.$$

Then $\operatorname{span}\mathcal{P}^\chi_\mathfrak{g} = U(\mathfrak{g}_-)^{-\chi}$. Write $\mathcal{M} = (\mathbb{Z}_+)^{\Delta_+}$ for the set of all tuples indexed by Δ_+, with entries in $\mathbb{Z}_+$, only finitely many of which may be non-zero. For $\chi \in \mathcal{Q}_+$, let

$$\mathcal{M}_\chi = \left\{ M \in \mathcal{M} \middle| \sum_{\alpha \in \Delta_+} M_\alpha \alpha = \chi \right\}.$$

For $M \in \mathcal{M}$, write $|M| = \sum_{\alpha\in\Delta_+} M_\alpha$, and let

$$\mathcal{P}_{\mathfrak{g}}^M = \{\lambda \in \mathcal{P}_{\mathfrak{g}} \,\big|\, |\lambda^\alpha| = M_\alpha \text{ for all } \alpha \in \Delta_+\}.$$

Then $\mathcal{P}_{\mathfrak{g}}^\chi = \bigsqcup_{M\in\mathcal{M}_\chi} \mathcal{P}_{\mathfrak{g}}^M$ for any $\chi \in \mathcal{Q}_+$. Thus,

$$\mathrm{d}_{\mathfrak{g}}^\chi := \sum_{\lambda\in\mathcal{P}_{\mathfrak{g}}^\chi} |\lambda| = \sum_{M\in\mathcal{M}_\chi} |M| \big|\mathcal{P}_{\mathfrak{g}}^M\big|.$$

Finally, if the triangular decomposition is non-degenerately paired, let $\mathbf{h}(M) = \prod_{\alpha\in\Delta_+} \mathbf{h}(\alpha)^{M_\alpha}$ for $M \in \mathcal{M}$.

3.5 Partitions for $\mathfrak{g} \otimes C$ and $\mathfrak{g} \otimes C_0$

Let $\mathfrak{g}$ denote a Lie algebra with triangular decomposition. Then $\mathfrak{g}_-$ has a basis parameterized by $\mathcal{B} = \mathcal{B}_{\mathfrak{g}}$. The $\Bbbk$-Lie algebra $\mathfrak{g} \otimes C$ has a basis $\mathrm{y} \otimes v$ parameterized by $\mathcal{B}_{\mathfrak{g}\otimes C} = \mathcal{B}_{\mathfrak{g}} \times \{1, \dots, l\}$,

$$\mathcal{B}_{\mathfrak{g}\otimes C} \ni (\gamma, i) \quad \leftrightarrow \quad \mathrm{y}(\gamma) \otimes v_i \in \mathfrak{g}^{-\mathrm{r}(\gamma)} \otimes \Bbbk v_i.$$

Recall, from Example 10, that $\mathfrak{g} \otimes C$ shares Δ_+ and $\mathcal{Q}_+$ with $\mathfrak{g}$. Thus it is sensible to define

$$\mathrm{r} : \mathcal{B}_{\mathfrak{g}\otimes C} \to \Delta_+, \quad (\gamma, i) \mapsto \mathrm{r}(\gamma).$$

Write $\mathcal{P}_{\mathfrak{g}\otimes C}$ for the set of all multisets with entries in $\mathcal{B}_{\mathfrak{g}\otimes C}$. Extend r to a map $\mathcal{P}_{\mathfrak{g}\otimes C} \to \mathcal{Q}_+$, and define the sets $\mathcal{P}_{\mathfrak{g}\otimes C}^\chi$ for $\chi \in \mathcal{Q}_+$ and $\mathcal{P}_{\mathfrak{g}\otimes C}^M$ for $M \in \mathcal{M}$, as per Sect. 3.4. The set $\mathcal{P}_{\mathfrak{g}\otimes C_0}$ and its subsets are defined in the same way, replacing C by C_0 in the above.

Given $\lambda \in \mathcal{P}_{\mathfrak{g}\otimes C}$ and $1 \le i \le l$, define $\lambda^i \in \mathcal{P}_{\rho_i(\mathfrak{g})}$ by

$$\lambda^i = \{\gamma \,|\, (\gamma, i) \in \lambda\},$$

counting multiplicities. This defines a bijection

$$\mathcal{P}_{\mathfrak{g}\otimes C} \leftrightarrow \mathcal{P}_{\rho_1(\mathfrak{g})} \times \cdots \times \mathcal{P}_{\rho_l(\mathfrak{g})}, \quad \lambda \leftrightarrow (\lambda^1, \dots, \lambda^l).$$

For every $M \in \mathcal{M}$, this restricts to a bijection

$$\mathcal{P}_{\mathfrak{g}\otimes C}^M \leftrightarrow \bigsqcup_{M^i \vdash M} \mathcal{P}_{\rho_1(\mathfrak{g})}^{M^1} \times \cdots \times \mathcal{P}_{\rho_l(\mathfrak{g})}^{M^l}, \tag{14}$$

where, here and henceforth, $M^i \vdash M$ indicates indexing over all l-tuples $(M^i) \in \mathcal{M}^l$ such that $\sum_{i=1}^l M^i = M$.

3.6 The Shapovalov Determinant

For any $\chi \in \mathcal{Q}_+$, write

$$\det \mathbf{F}_{\mathfrak{g}}^{\chi} = \det\bigl(\mathbf{F}_{\mathfrak{g}}^{\chi}\bigl(\mathrm{y}(\lambda), \mathrm{y}(\mu)\bigr)\bigr)_{\lambda,\mu \in \mathcal{P}_{\mathfrak{g}}^{\chi}}$$

for the *Shapovalov determinant* at χ. By Proposition 12,

$$\bigl\langle \Lambda, \det \mathbf{F}_{\mathfrak{g}}^{\chi} \bigr\rangle = 0 \tag{15}$$

if and only if the maximal submodule of the Verma module $\mathrm{M}(\Lambda)$ has a non-trivial $\Lambda|_{\mathfrak{h}_0} - \chi$ weight space for any $\Lambda \in \mathfrak{h}^*$. In particular, $\mathrm{M}(\Lambda)$ is reducible precisely when equation (15) holds for some $\chi \in \mathcal{Q}_+$.

For $f \in \mathrm{S}(\mathfrak{h})$, write $f\langle i \rangle$ for the component of homogeneous degree i in $\mathfrak{h}$, for i a non-negative integer. Write $\max_{\mathfrak{h}} f = f\langle \deg_{\mathfrak{h}} f \rangle$. For a proof of the following lemma, which is due to Shapovalov [5], see [6] or [3].

Lemma 16 *Let* $(\mathfrak{g}, \mathfrak{h}_0, \mathfrak{h}, \mathfrak{g}_+, \omega)$ *be a Lie algebra with triangular decomposition. Suppose that* $\lambda, \mu \in \mathcal{P}$. *Then*

(i) $\deg_{\mathfrak{h}} \boldsymbol{F}(\mathrm{y}(\lambda), \mathrm{y}(\mu)) \le |\lambda|, |\mu|$;
(ii) *if* $|\lambda| = |\mu|$ *and* $|\lambda^{\alpha}| \neq |\mu^{\alpha}|$ *for some* $\alpha \in \Delta_+$, *then* $\deg_{\mathfrak{h}} \boldsymbol{F}(\mathrm{y}(\lambda), \mathrm{y}(\mu)) < |\lambda| = |\mu|$;
(iii) *if* $|\lambda| = |\mu|$ *and* $|\lambda^{\alpha}| = |\mu^{\alpha}| =: m_{\alpha}$ *for all* $\alpha \in \Delta_+$, *then*

$$\boldsymbol{F}(\mathrm{y}(\lambda), \mathrm{y}(\mu))\langle |\lambda| \rangle = \prod_{\alpha \in \Delta_+} \sum_{\tau \in \mathrm{Sym}(m_{\alpha})} \prod_{1 \le j \le m_{\alpha}} \bigl[\omega\bigl(\mathrm{y}\bigl(\lambda^{\alpha}_{\tau(j)}\bigr)\bigr), \mathrm{y}\bigl(\mu^{\alpha}_{j}\bigr)\bigr],$$

where for each $\alpha \in \Delta_+$, $(\lambda^{\alpha}_{j})_{1 \le j \le m_{\alpha}}$ *and* $(\mu^{\alpha}_{j})_{1 \le j \le m_{\alpha}}$ *are any fixed enumerations of* λ^{α} *and* μ^{α}, *respectively.*

The following proposition is immediate from Lemma 16.

Proposition 17

(i) $\deg_{\mathfrak{h}} \det \boldsymbol{F}^{\chi} \le \mathrm{d}_{\mathfrak{g}}^{\chi}$ *and* $\deg_{\mathfrak{h}} \det \boldsymbol{F}|_{\mathrm{span}\,\mathcal{P}^M} \le |M||\mathcal{P}^M|$ *for any* $M \in \mathcal{M}$.
(ii) $(\det \boldsymbol{F}^{\chi})\langle \mathrm{d}_{\mathfrak{g}}^{\chi} \rangle = \prod_{M \in \mathcal{M}_{\chi}} (\det \boldsymbol{F}|_{\mathrm{span}\,\mathcal{P}^M})\langle |M||\mathcal{P}^M| \rangle$.
(iii) *If* $\mathfrak{g}$ *is non-degenerately paired, then*

$$\max_{\mathfrak{h}} \det \boldsymbol{F}|_{\mathrm{span}\,\mathcal{P}^M} = \mathbf{h}(M)^{|\mathcal{P}^M|}$$

for any $M \in \mathcal{M}$.

4 Contraction of the Shapovalov Determinant

Throughout this section, $\mathfrak{g}$ denotes a $\Bbbk$-Lie algebra with triangular decomposition. In this section, the contraction of $\mathfrak{g} \otimes C$ to $\mathfrak{g} \otimes C_0$, described in Sect. 2, is employed to express the Shapovalov determinant $\det \mathbf{F}^{\chi}_{\mathfrak{g}\otimes C_0}$ in terms of $\det \mathbf{F}^{\chi}_{\mathfrak{g}\otimes C}$ (see Proposition 20). The remainder of the section studies $\det \mathbf{F}^{\chi}_{\mathfrak{g}\otimes C}$ when $\mathfrak{g}$ is non-degenerately paired. This then leads to the derivation of a closed formula for $\det \mathbf{F}^{\chi}_{\mathfrak{g}\otimes C_0}$ (see Theorem 32). For any vector space V and $k \geq 0$, let $\mathrm{S}^k(V)$ denote the homogeneous degree-k subspace of $\mathrm{S}(V)$.

Lemma 18

(i) *Suppose that* V_1, V_2 *are finite-dimensional vector spaces and* $\varphi_i \in \operatorname{End} V_i$ *for* $i = 1, 2$. *Then*

$$\det \varphi_1 \otimes \varphi_2 = (\det \varphi_1)^{\dim V_2} (\det \varphi_2)^{\dim V_1}.$$

(ii) *Suppose that* V *is a finite-dimensional vector space and that* $\varphi \in \operatorname{End} V$. *Then*

$$\det \mathrm{S}^k(\varphi) = (\det \varphi)^{k \cdot \frac{\dim \mathrm{S}^k(V)}{\dim V}},$$

where $\mathrm{S}^k(\varphi)$ *denotes the endomorphism of* $\mathrm{S}^k(V)$ *induced by* φ.

For any $\chi \in \mathcal{Q}_+$, write $\theta_z|_{\mathrm{U}(\mathfrak{g}_-\otimes C)^{-\chi}}$ for the vector space endomorphism of $\mathrm{U}(\mathfrak{g}_- \otimes C)^{-\chi}$ induced by $\theta_z \in \operatorname{End}_{\Bbbk} \mathbf{V}_l$.

Lemma 19 *For any* $\chi \in \mathcal{Q}_+$,

$$\det \theta_z|_{\mathrm{U}(\mathfrak{g}_-\otimes C)^{-\chi}} = (\det \theta_z)^{l^{-1} \cdot \mathrm{d}^{\chi}_{\mathfrak{g}\otimes C}}.$$

Proof By the PBW Theorem, there is an isomorphism of vector spaces

$$\mathrm{U}(\mathfrak{g}_- \otimes C)^{-\chi} \cong \bigoplus_{M \in \mathcal{M}_\chi} \bigotimes_{\alpha \in \Delta_+} \mathrm{S}^{M_\alpha}\big(\mathfrak{g}^{-\alpha} \otimes C\big).$$

Thus, by Lemma 18, $\det \theta_z|_{\mathrm{U}(\mathfrak{g}_-\otimes C)^{-\chi}}$ can be written as

$$\begin{aligned} &\prod_{M \in \mathcal{M}_\chi} \prod_{\alpha \in \Delta_+} \Big(\big((\det \theta_z)^{\dim \mathfrak{g}^\alpha}\big)^{M_\alpha \cdot \frac{\dim \mathrm{S}^{M_\alpha}(\mathfrak{g}^{-\alpha}\otimes C)}{l \cdot \dim \mathfrak{g}^\alpha}} \Big)^{\prod_{\beta \neq \alpha} \dim \mathrm{S}^{M_\beta}(\mathfrak{g}^{-\beta}\otimes C)} \\ &= (\det \theta_z)^{l^{-1} \sum_{M \in \mathcal{M}_\chi} |M| \prod_{\beta \in \Delta_+} \dim \mathrm{S}^{M_\beta}(\mathfrak{g}^{-\beta}\otimes C)} \\ &= (\det \theta_z)^{l^{-1} \cdot \mathrm{d}^{\chi}_{\mathfrak{g}\otimes C}}. \end{aligned}$$

□

Proposition 20 *For any* $\chi \in \mathcal{Q}_+$,

$$\det \boldsymbol{F}^{\chi}_{\mathfrak{g}\otimes C_0} = \Big[(\det \theta_z)^{2 \cdot l^{-1} \cdot \mathrm{d}^{\chi}_{\mathfrak{g}\otimes C}} \cdot \big(\theta_z^{-1} \circ \iota\big)\big(\det \boldsymbol{F}^{\chi}_{\mathfrak{g}\otimes C}\big) \Big]_{z=0}.$$

Proof The enveloping algebra of the $\Bbbk[z, z^{-1}]$-Lie algebra $\mathfrak{g} \otimes_{\Bbbk} \mathbf{C}_z$ has a basis consisting of the images under ι of the PBW monomials for $\mathfrak{g}$ (see [1]). Moreover, $\mathrm{U}(\mathfrak{g}_- \otimes \mathbf{C}_z)$ carries a Shapovalov form $\mathbf{F}_{\mathfrak{g}\otimes\mathbf{C}_z}$, defined as per Sect. 3 (see [3] for the general construction). The determinants of a bilinear form, calculated with respect to two different choices of bases, are related by the square of determinant of the change of basis. Write $\det_u \mathbf{F}^\chi$ for the determinant of the Shapovalov form calculated with respect to the basis of PBW monomials induced by an ordered basis u of the negative part of the triangular decomposition. The basis $v = (v_i)_{1\le i\le l}$ of $\mathbf{V}_l$ defines an ordered basis $\mathrm{y} \otimes v$ of $\mathfrak{g}_- \otimes \mathbf{V}_l$ where $\mathrm{y} = (\mathrm{y}(\gamma))_{\gamma\in\mathcal{B}}$ is the fixed ordered basis for $\mathfrak{g}_-$. The $\Bbbk[z, z^{-1}]$-module $\mathfrak{g}_- \otimes \mathbf{V}_l \otimes \Bbbk[z, z^{-1}]$ has bases $\mathrm{y} \otimes \iota(v)$ and $\mathrm{y} \otimes T$, where $T = (T_i(z))_{1\le i\le l}$. By Proposition 7, part (ii),

$$v_i = \sum_{j=1}^{l} \left(\theta_z^t\right)_{i,j} T_j(z), \quad 1 \le i \le l.$$

Thus, by Lemma 19,

$$\det\nolimits_{\mathrm{y}\otimes\iota(v)} \mathbf{F}^\chi_{\mathfrak{g}\otimes\mathbf{C}_z} = (\det\theta_z)^{2\cdot l^{-1}\cdot \mathrm{d}^\chi_{\mathfrak{g}\otimes C}} \det\nolimits_{\mathrm{y}\otimes T} \mathbf{F}^\chi_{\mathfrak{g}\otimes\mathbf{C}_z}. \tag{21}$$

$\theta_z : \mathfrak{g} \otimes \mathbf{C}_z \to \mathfrak{g} \otimes C \otimes \Bbbk[z, z^{-1}]$ is an isomorphism of $\Bbbk[z, z^{-1}]$-Lie algebras, and by Proposition 7, part (ii), $\theta_z : \mathrm{y} \otimes T \mapsto \mathrm{y} \otimes \iota(v)$. Thus,

$$\theta_z\left(\det\nolimits_{\mathrm{y}\otimes T} \mathbf{F}^\chi_{\mathfrak{g}\otimes\mathbf{C}_z}\right) = \det\nolimits_{\mathrm{y}\otimes\iota(v)} \mathbf{F}^\chi_{\mathfrak{g}\otimes C\otimes\Bbbk[z,z^{-1}]}. \tag{22}$$

The Lie algebra $\mathfrak{g} \otimes C \otimes \Bbbk[z, z^{-1}]$ is constructed from $\mathfrak{g} \otimes C$ by extension of scalars, and ι is the associated injection. Thus,

$$\det\nolimits_{\mathrm{y}\otimes\iota(v)} \mathbf{F}^\chi_{\mathfrak{g}\otimes C\otimes\Bbbk[z,z^{-1}]} = \iota\left(\det\nolimits_{\mathrm{y}\otimes v} \mathbf{F}^\chi_{\mathfrak{g}\otimes C}\right). \tag{23}$$

Finally, by (9),

$$\det\nolimits_{\mathrm{y}\otimes v} \mathbf{F}^\chi_{\mathfrak{g}\otimes C_0} = \left[\det\nolimits_{\mathrm{y}\otimes\iota(v)} \mathbf{F}^\chi_{\mathfrak{g}\otimes\mathbf{C}_z}\right]_{z=0}. \tag{24}$$

Beginning with (24), the claim follows from substitutions (21), (22) and (23). □

Proposition 25 *Suppose that* $\mathfrak{g}$ *is non-degenerately paired. Then*

$$\max_{\mathfrak{h}} \det \boldsymbol{F}^\chi_{\mathfrak{g}\otimes C} = \prod_{M\in\mathcal{M}_\chi} \prod_{M^i\vdash M} \prod_{i=1}^{l} \rho_i\left(\mathbf{h}\left(M^i\right)\right)^{\prod_{j=1}^{l} |\mathcal{P}_{\mathfrak{g}}^{M^j}|}.$$

Proof Let $M \in \mathcal{M}$. The bijection (14) defines a vector space isomorphism

$$\operatorname{span} \mathcal{P}^M_{\mathfrak{g}\otimes C} = \bigoplus_{M^i\vdash M} \bigotimes_{i=1}^{l} \rho_i\left(\operatorname{span} \mathcal{P}_{\mathfrak{g}}^{M^i}\right). \tag{26}$$

Suppose that $\lambda, \mu \in \mathcal{P}^M_{\mathfrak{g}\otimes C}$ and that $|\lambda^i| \neq |\mu^i|$ for some i. Then by Lemma 16, part (ii),

$$\deg_{\mathfrak{h}} \mathbf{F}_{\mathfrak{g}}\big(\mathrm{y}(\lambda^i), \mathrm{y}(\mu^i)\big) < \big|\lambda^i\big|, \big|\mu^i\big|.$$

Thus, since the v_i are orthogonal idempotents in C,

$$\deg_{\mathfrak{h}} \mathbf{F}_{\mathfrak{g}\otimes C}\big((\mathrm{y}\otimes v)(\lambda), (\mathrm{y}\otimes v)(\mu)\big) < |\lambda| = |\mu| = |M|.$$

Hence, using (26),

$$\det \mathbf{F}|_{\operatorname{span}\mathcal{P}^M_{\mathfrak{g}\otimes C}}\big\langle |M| \big|\mathcal{P}^M_{\mathfrak{g}\otimes C}\big|\big\rangle = \Bigg(\prod_{M^i \vdash M} \det \mathbf{F}|_{\bigotimes_{i=1}^l \rho_i(\operatorname{span}\mathcal{P}^{M^i}_{\mathfrak{g}})}\Bigg)\big\langle |M| \big|\mathcal{P}^M_{\mathfrak{g}\otimes C}\big|\big\rangle, \quad (27)$$

since by Proposition 17, part (ii), this is the highest-degree term. Now

$$\mathbf{F}_{\mathfrak{g}\otimes C}|_{\bigotimes_{i=1}^l \rho_i(\operatorname{span}\mathcal{P}^{M^i}_{\mathfrak{g}})} = \bigotimes_{i=1}^{l} \mathbf{F}_{\mathfrak{g}\otimes C}|_{\rho_i(\operatorname{span}\mathcal{P}^{M^i}_{\mathfrak{g}})},$$

since the v_i are orthogonal idempotents in C. Thus,

$$\det \mathbf{F}_{\mathfrak{g}\otimes C}|_{\bigotimes_{i=1}^l \rho_i(\operatorname{span}\mathcal{P}^{M^i}_{\mathfrak{g}})} = \prod_{i=1}^{l}\big(\det \mathbf{F}_{\mathfrak{g}\otimes C}|_{\rho_i(\operatorname{span}\mathcal{P}^{M^i}_{\mathfrak{g}})}\big)^{\prod_{j\neq i}|\mathcal{P}^{M^j}_{\mathfrak{g}}|}, \quad (28)$$

by Lemma 18, part (ii). By part (iii) of Proposition 17,

$$\det \mathbf{F}_{\mathfrak{g}\otimes C}|_{\rho_i(\operatorname{span}\mathcal{P}^{M^i}_{\mathfrak{g}})} = \rho_i\big(\det \mathbf{F}_{\mathfrak{g}}|_{\operatorname{span}\mathcal{P}^{M^i}_{\mathfrak{g}}}\big) = \rho_i\big(\mathbf{h}(M^i)\big)^{|\mathcal{P}^{M^i}_{\mathfrak{g}}|}. \quad (29)$$

Thus, substituting (29) into (28), we have

$$\det \mathbf{F}_{\mathfrak{g}\otimes C}|_{\bigotimes_{i=1}^l \rho_i(\operatorname{span}\mathcal{P}^{M^i}_{\mathfrak{g}})} = \Bigg(\prod_{i=1}^{l} \rho_i\big(\mathbf{h}(M^i)\big)\Bigg)^{\prod_{j=1}^l |\mathcal{P}^{M^j}_{\mathfrak{g}}|}. \quad (30)$$

Finally, by Proposition 17 part (ii),

$$\begin{aligned}\big(\det \mathbf{F}^{\chi}_{\mathfrak{g}\otimes C}\big)\big\langle \mathrm{d}^{\chi}_{\mathfrak{g}\otimes C}\big\rangle &= \prod_{M\in\mathcal{M}_\chi} \big(\det \mathbf{F}|_{\operatorname{span}\mathcal{P}^M_{\mathfrak{g}\otimes C}}\big)\big\langle |M| \big|\mathcal{P}^M_{\mathfrak{g}\otimes C}\big|\big\rangle \\ &= \prod_{M\in\mathcal{M}_\chi}\Bigg(\prod_{M^i\vdash M} \det \mathbf{F}_{\mathfrak{g}\otimes C}|_{\bigotimes_{i=1}^l \rho_i(\operatorname{span}\mathcal{P}^{M^i}_{\mathfrak{g}})}\Bigg)\big\langle |M| \big|\mathcal{P}^M_{\mathfrak{g}\otimes C}\big|\big\rangle\end{aligned}$$

(by (27))

$$= \prod_{M\in\mathcal{M}_\chi}\left(\prod_{M^i\vdash M}\prod_{i=1}^{l}\rho_i\big(\mathbf{h}\big(M^i\big)\big)^{\prod_{j=1}^{l}|\mathcal{P}_{\mathfrak{g}}^{M^j}|}\right)^{\langle|M||\mathcal{P}_{\mathfrak{g}\otimes C}^{M}|\rangle}$$

(by (30))

$$= \prod_{M\in\mathcal{M}_\chi}\prod_{M^i\vdash M}\prod_{i=1}^{l}\rho_i\big(\mathbf{h}\big(M^i\big)\big)^{\prod_{j=1}^{l}|\mathcal{P}_{\mathfrak{g}}^{M^j}|}.$$

□

If f is a polynomial in z^{-1}, write $\max_{z^{-1}} f$ for the component of maximal degree in z^{-1}.

Proposition 31 *Suppose that* $\mathfrak{g}$ *is non-degenerately paired. Then for any* $\chi \in \mathcal{Q}_+$,

$$\max_{z^{-1}}\big(\theta_z^{-1}\circ\iota\big)\big(\det \boldsymbol{F}_{\mathfrak{g}\otimes C}^{\chi}\big) = z^{-(l-1)\cdot \mathrm{d}_{\mathfrak{g}\otimes C}^{\chi}}\prod_{M\in\mathcal{M}_\chi}(\iota\circ\rho_l)\big(\mathbf{h}(M)\big)^{|\mathcal{P}_{\mathfrak{g}\otimes C}^{M}|}.$$

Proof For any $1 \le i \le l$, notice that

$$\max_{z^{-1}}\big(\theta_z^{-1}\circ\iota\circ\rho_i\big)(1) = \max_{z^{-1}} T_i(z) = -\sigma_i^{-1} z^{-(l-1)}\epsilon_0\big(\hat{\zeta}_i\big)v_l,$$

by (5). Thus $\max_{z^{-1}}(\theta_z^{-1}\circ\iota\circ\rho_i) = z^{-(l-1)}(\iota\circ\rho_l)$, up to a non-zero scalar in $\Bbbk$, and in particular, is independent of i. Thus, using firstly that $T_i(z)$ is polynomial in z^{-1}, we have

$$\begin{aligned}
&\max_{z^{-1}}\big(\theta_z^{-1}\circ\iota\big)\big(\det \mathbf{F}_{\mathfrak{g}\otimes C}^{\chi}\big)\\
&\quad= \max_{z^{-1}}\big(\theta_z^{-1}\circ\iota\big)\Big(\max_{\mathfrak{h}}\det \mathbf{F}_{\mathfrak{g}\otimes C}^{\chi}\Big)\\
&\quad= \max_{z^{-1}}\big(\theta_z^{-1}\circ\iota\big)\Bigg(\prod_{M\in\mathcal{M}_\chi}\prod_{M^i\vdash M}\prod_{i=1}^{l}\rho_i\big(\mathbf{h}\big(M^i\big)\big)^{\prod_{j=1}^{l}|\mathcal{P}_{\mathfrak{g}}^{M^j}|}\Bigg) \quad \text{(by Proposition 25)}\\
&\quad= \prod_{M\in\mathcal{M}_\chi}\prod_{M^i\vdash M}\left(\prod_{i=1}^{l} z^{-(l-1)|M^i|}\cdot(\iota\circ\rho_l)\big(\mathbf{h}\big(M^i\big)\big)\right)^{\prod_{j=1}^{l}|\mathcal{P}_{\mathfrak{g}}^{M^j}|}\\
&\quad= \prod_{M\in\mathcal{M}_\chi}\Big(z^{-(l-1)\sum_{M^i\vdash M}|M|}\cdot(\iota\circ\rho_l)(\mathbf{h}(M))\Big)^{\sum_{M^i\vdash M}\prod_{j=1}^{l}|\mathcal{P}_{\mathfrak{g}}^{M^j}|}\\
&\quad= z^{-(l-1)\cdot\mathrm{d}_{\mathfrak{g}\otimes C}^{\chi}}\prod_{M\in\mathcal{M}_\chi}(\iota\circ\rho_l)\big(\mathbf{h}(M)\big)^{|\mathcal{P}_{\mathfrak{g}\otimes C}^{\chi}|}.
\end{aligned}$$

□

Theorem 32 *Suppose that* $\mathfrak{g}$ *is non-degenerately paired. Then*

$$\det \boldsymbol{F}^{\chi}_{\mathfrak{g}\otimes C_0} = \prod_{M \in \mathcal{M}_\chi} \rho_l\big(\mathbf{h}(M)\big)^{|\mathcal{P}^M_{\mathfrak{g}\otimes C_0}|}$$

for any $\chi \in \mathcal{Q}_+$.

Proof By Proposition 20 and Proposition 7 part (i),

$$\begin{aligned}\det \mathbf{F}^{\chi}_{\mathfrak{g}\otimes C_0} &= \left[z^{(l-1)\mathrm{d}^{\chi}_{\mathfrak{g}\otimes C}} \cdot \left(\theta_z^{-1} \circ \iota\right)\left(\det \mathbf{F}^{\chi}_{\mathfrak{g}\otimes C}\right)\right]_{z=0} \\ &= \left[z^{(l-1)\mathrm{d}^{\chi}_{\mathfrak{g}\otimes C}} \cdot \max_{z^{-1}} \left(\theta_z^{-1} \circ \iota\right)\left(\det \mathbf{F}^{\chi}_{\mathfrak{g}\otimes C}\right)\right]_{z=0} \\ &= \left[\prod_{M \in \mathcal{M}_\chi} (\iota \circ \rho_l)\big(\mathbf{h}(M)\big)^{|\mathcal{P}^M_{\mathfrak{g}\otimes C}|}\right]_{z=0},\end{aligned}$$

where the last equality holds by Proposition 31. The claim now follows, since

$$\left[(\iota \circ \rho_l)(\cdot)\right]_{z=0} = \rho_l(\cdot)$$

and $|\mathcal{P}^M_{\mathfrak{g}\otimes C}| = |\mathcal{P}^M_{\mathfrak{g}\otimes C_0}|$. □

Theorem 3 is an immediate corollary of Theorem 32 and Proposition 12.

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